Problem 1: Uncertainty Principle for Angular Momentum Eigenstates

Consider a wave function for the angular variables that is equal to one of the spherical harmonics, $Y_l^m(\theta, \phi)$.

(a) Show that in this state $\langle \hat{L}_x \hat{L}_y \rangle = \langle \hat{L}_z \hat{L}_z \rangle = \hbar m \langle \hat{L}_z \rangle$ (Hint: Remember, $\hat{L}_z$ is a Hermitian operator).

(b) Use the commutator $[\hat{L}_y, \hat{L}_z]$ to show that $\langle \hat{L}_x \rangle = 0$.

(c) Follow a similar procedure to show that $\langle \hat{L}_y \rangle = 0$.

(d) Assuming by symmetry $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle$, show that $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \frac{\hbar^2 (l(l+1) - m^2)}{2}$.

Hint: Consider $\hat{L}^2$.

(e) Show that this state obeys uncertainty relation, $\Delta L_x \Delta L_y \geq \frac{1}{2} |\langle \hat{L}_z \rangle|$.

Problem 2: The Finite Spherical Well

Consider a spherically symmetric potential, $V(r) = \begin{cases} -V_0 & 0 < r < a \\ 0 & r > a \end{cases}$. Along the radial coordinate, due to the boundary condition at $r=0$, this is just the half-finite well we studied in Problem Sets 5 and 6.
(a) For \( E < 0 \), the solutions to the T.I.S.E. are bound states. Let \( E = -E_b \). Making the ansatz for the stationary state wave functions \( \psi_{E,l,m}(r, \theta, \phi) = R_{E,l}(r)Y^m_l(\theta, \phi) \), show that the radial function must have the form,

\[
R_{E,l}(r) = \begin{cases} 
A \ j_l(k_r) & 0 < r < a \\
A \ j_l(k_a) \ h^{(1)}_l(ikr) & r > a
\end{cases},
\]

where \( k_l = \sqrt{\frac{2m}{\hbar^2}(V_0 - E_b)} \), \( \kappa = \sqrt{\frac{2m}{\hbar^2}E_b} \).

How would you determine \( A \)?

(b) Show that the binding energies are determined by the transcendental equation

\[
\left( \frac{d}{dr} \left( r j_l(k_r) \right) \right)_{r=a} = \left( \frac{d}{dr} \left( r h^{(1)}_l(ikr) \right) \right)_{r=a}.
\]

Does this reduce to the expected solution of \( s \)-states (i.e. \( l = 0 \)).

(c) Now consider the unbound states. We seek the scattering phase shift for the asymptotic incoming and outgoing partial waves, as discussed in Lecture 21.

\[
V_{eff}^{(1)} \quad \rightarrow \quad -ih^{(2)}_l(kr) \quad \rightarrow \quad -ie^{i\theta} h^{(1)}_l(kr)
\]

Show that the phase satisfies the equation

\[
\left( \frac{r j_l(qr)}{\frac{d}{dr} [r j_l(qr)]} \right)_{r=a} = \left( \frac{r \left( \cos(\delta_l/2) j_l(kr) - \sin(\delta_l/2)n_l(kr) \right)}{\frac{d}{dr} \left[ r \left( \cos(\delta_l/2) j_l(ka) - \sin(\delta_l/2)n_l(ka) \right) \right]} \right)_{r=a},
\]

where \( k = \sqrt{\frac{2m}{\hbar^2}E} \) and \( q = \sqrt{\frac{2m}{\hbar^2}(E + V_0)} \).

Check that this limits to the expected result for \( s \)-wave (\( l=0 \)).