Problem 1: 1D, 2D vs. 3D geometries
(a) Consider a particle subject to a two-dimensional infinite square well potential

\[ V(x, y) = \begin{cases} 
0, & 0 < x < a_x, 0 < y < a_y \\
\infty, & \text{otherwise} 
\end{cases} \]

and free in the z-direction. Show that the energy spectrum is

\[ E_{n_x, n_y}(k_z) = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right), \]

where \( k_{nx} = n_x \frac{\pi}{a_x}, \) \( k_{ny} = n_y \frac{\pi}{a_y}, \) \( n_x, n_y = 1, 2, \ldots, \) \( k_z \) arbitrary.

What are the eigenfunctions? What is their degeneracy?

(b) Now consider a 3D well with \( a_z \ll a_x, a_y.\) Let us choose \( a_x = a_y = a \) and \( a = 10a_z.\)

This is a kind of “slab geometry”. What is the energy spectrum? What is their degeneracy. Make a sketch of the energy levels. Please comment.

(c) Now consider a 3D well with \( a_z \gg a_x, a_y.\) Let us choose \( a_x = a_y = a \) and \( a = a_z / 10.\)

This is a kind of “wire geometry”. What is the energy spectrum? What is their degeneracy. Make a sketch of the energy levels. Please comment.

(d) Suppose you wanted to “engineer” a situation to study a gas in 2D or a gas in 1D. Explain how you might do this. Note, the temperature will play an important role.

Problem 2: The rigid rotator
Consider a dumbbell model of a diatomic molecule, with two masses attached rigidly to a massless rod of length \( d.\)

(a) Assuming \( d \) cannot change, and its center of mass does not change, show that the Hamiltonian is

\[ \hat{H} = \frac{\hat{L}^2}{2I^2} \]

where \( \hat{L}^2 \) is the squared angular momentum and \( I \) is the moment of inertia of the masses for rotation perpendicular to the dumbbell.
(b) What are the energy levels of the system? What is their degeneracy? What are the energy eigenfunctions? Sketch a level diagram.

(c) Diatomic nitrogen, with \( d = 100 \) pm. Suppose a quantum jump occurs from level denoted by quantum number \( l+1 \) to \( l \). What is the wavelength of the emitted?

(d) Suppose you measured the spectrum of emitted light between different transitions. Explain how you would use it to measure the moment of inertia of the molecule.

**Problem 3: Exercises with angular momentum**

(a) Show that \( [\hat{L}_x, \hat{y}] = i\hbar \hat{z} \) and \( [\hat{L}_z, \hat{p}_y] = i\hbar \hat{p}_z \). Write down their cyclic permutations. These express the fact that position and momentum are vectors and behave a certain way under rotation.

(b) Show that \( [\hat{L}_x, \hat{r}^2] = 0 \) and \( [\hat{L}_z, \hat{p}^2] = 0 \), where \( \hat{L}_i \) is any component of angular momentum. These express the fact that \( \hat{r}^2 \) and \( \hat{p}^2 \) are scalars that are invariant under rotation. From these show that for a central potential, \( \{\hat{H}, \hat{L}_x, \hat{L}_z\} \) form a set of mutually commuting operators.

Consider the wave function \( \psi(x,y,z) = \frac{15}{8\pi} \frac{(y - iz)x}{r^2} \).

(c) Show that this is only a function of \( \theta \) and \( \phi \), and normalized on the sphere.

(d) Show that this an eigenfunction of \( \hat{L}^2 \) and \( \hat{L}_z \). What are the eigenvalues?

(e) Express \( \psi \) as a superposition of spherical harmonics.

(f) If a measurement of \( \hat{L}_z \) is made, what values can be found and with what probabilities?