Can preparation operations be specified by classical facts,
independent of the input quantum state?

Carlton M. Caves

2007 November 6

Preparation operations

A preparation operation $\mathcal{A}$ is one that prepares the same output state σ regardless of input, i.e.,

$$\mathcal{A}(\rho) = p(\rho)\sigma ,$$

for all ρ for which

$$p(\rho) = \mathrm{tr}\big(\mathcal{A}(\rho)\big) \neq 0 .$$

Writing $\mathcal{A}$ in terms of a Kraus decomposition,

$$\mathcal{A} = \sum_\alpha A_\alpha \odot A_\alpha^\dagger ,$$

we have

$$p(\rho) = \mathrm{tr}(E\rho) ,$$

where

$$E = \sum_\alpha A_\alpha^\dagger A_\alpha$$

is the POVM element corresponding to $\mathcal{A}$. Writing the eigendecomposition of

$$E = \sum_j \mu_j |g_j\rangle\langle g_j| ,$$

we see that the input states for which $p(\rho) = 0$ are those that are confined to the null subspace of E. In analyzing $\mathcal{A}$, we can confine our attention to states that lie in the support of E.

If we write the eigendecomposition of

$$\sigma = \sum_k \lambda_k |f_k\rangle\langle f_k| ,$$

it is easy to see that $\mathcal{A}$ has the form

$$\mathcal{A} = \sum_{j,k} \lambda_k \mu_j |f_k\rangle\langle g_j| \odot |g_j\rangle\langle f_k| = \sum_{j,k} \sqrt{\sigma}\, |f_k\rangle\langle g_j|\sqrt{E} \odot \sqrt{E}|g_j\rangle\langle f_k|\sqrt{\sigma} .$$

Thus one set of Kraus operators for $\mathcal{A}$ is $\{\sqrt{\sigma}\,|f_k\rangle\langle g_j|\sqrt{E}\}$.

If $\mathcal{A}$ is trace preserving, it is a *deterministic* preparation operation, having $p(\rho) = 1$ for all inputs, and if $\mathcal{A}$ is trace decreasing, it is a *stochastic* preparation operation, which only works part of the time. A preparation operation is deterministic iff $E = I$. A stochastic preparation operation that prepares a pure state can always be thought of as a measurement in an arbitrary orthonormal basis followed by a conditional unitary that maps the measured state to the desired pure state.

Measurement models

Now let's consider measurement models for $\mathcal{A}$, i.e.,

$$\mathcal{A}(\rho) = \mathrm{tr}_A(\Pi_A U \rho \otimes \Sigma\, U^\dagger) = \sum_{l,m} \underbrace{\sqrt{\eta_l}\langle F_m|U|G_l\rangle}_{= A_\alpha}\, \rho\, \underbrace{\langle G_l|U^\dagger|F_m\rangle\sqrt{\eta_l}}_{= A_\alpha^\dagger} ,$$

where

$$\Sigma = \sum_l \eta_l |G_l\rangle\langle G_l|$$

is the initial state of the ancilla (or apparatus) and

$$\Pi_A = \sum_m |F_m\rangle\langle F_m|$$

is an ancilla projector that projects onto subspace S_A. Our concern is whether the output state can be made independent of the input product state. Let's denote the system dimension by d and the ancilla dimension by D.

Measurement models: Deterministic preparation operations

Let's concentrate first on deterministic operations, for which $\Pi_A = I_A$. What we're interested in is having the measurement model produce system output state σ for every input *product** state. For this to happen, it must be true, for some product basis $|e_j\rangle \otimes |E_l\rangle$, that

$$|\beta_{j,l}\rangle = U|e_j\rangle \otimes |E_l\rangle = \sum_k \sqrt{\lambda_k}|f_k\rangle \otimes |F^k_{j,l}\rangle , \tag{1a}$$

where the last form is the Schmidt decomposition of $|\beta_{j,l}\rangle$, i.e.,

$$\langle F^k_{j,l}|F^{k'}_{j,l}\rangle = \delta_{kk'} , \tag{1b}$$

and where the output states $|\beta_{j,l}\rangle$, numbering dD, are an orthonormal basis for the joint system, i.e.,

$$\delta_{jj'}\delta_{ll'} = \langle \beta_{j,l}|\beta_{j',l'}\rangle = \sum_k \lambda_k \langle F^k_{j,l}|F^k_{j',l'}\rangle . \tag{1c}$$

Requirement (1) is not sufficient; in addition, we need to have that

$$U|\psi\rangle \otimes |\Psi\rangle = \sum_{j,l} a_j b_l |\beta_{j,l}\rangle = \sum_k \sqrt{\lambda_k}|f_k\rangle \otimes \left(\sum_{j,l} a_j b_l |F^k_{j,l}\rangle\right) \tag{2a}$$

* We might require the output state to be independent of input for *every* input state, not just product states, and requiring this would make the proof for stochastic preparation operations much simpler, but it would be far less convincing if this result depended on inputting entangled states.

marginalizes to σ for all input product states $|\psi\rangle \otimes |\Psi\rangle$, where

$$|\psi\rangle = \sum_j a_j |e_j\rangle \qquad \text{and} \qquad |\Psi\rangle = \sum_l b_l |E_l\rangle \ .$$

This means that for arbitrary choices of the expansion coefficients a_j and b_l, the states

$$\sum_{j,l} a_j b_l |F^k_{j,l}\rangle \ , \quad k = 1, \ldots, \mathrm{rank}(\sigma) \ , \tag{2b}$$

must be orthonormal. The orthonormality of the states (2b), which includes Eq. (1b), requires that

$$\langle F^k_{j,l}|F^{k'}_{j',l'}\rangle = \delta_{kk'}\delta_{jj'}\delta_{ll'} \ , \tag{3}$$

as we show below. The number of states $|F^k_{j,l}\rangle$ being $dD \times \mathrm{rank}(\sigma)$, Eq. (3) is clearly impossible. Notice that we did not need property (1c).

To show Eq. (3), we begin by defining joint system operators

$$M^{kk'} = \sum_{j,l,j',l'} |e_j\rangle \otimes |E_l\rangle \underbrace{\langle F^k_{j,l}|F^{k'}_{j',l'}\rangle}_{= M^{kk'}_{j,l;j',l'}} \langle e_{j'}| \otimes \langle E_{l'}| \ ,$$

whose matrix elements are the various inner-product matrices. The orthonormality of the states (2b) then becomes

$$\delta_{kk'} = \sum_{j,l,j',l'} a^*_j a_{j'} b^*_l b_{l'} \langle F^k_{j,l}|F^{k'}_{j',l'}\rangle = (\langle\psi| \otimes \langle\Psi|) M^{kk'} (|\psi\rangle \otimes |\Psi\rangle)$$

for all product states $|\psi\rangle \otimes |\Psi\rangle$. Since the space of joint operators has a product-operator basis and the operator space for each subsystem has a basis of rank-one projectors, $M^{kk'}$ is determined by this condition to be $M^{kk'} = \delta_{kk'}$, and from this Eq. (3) follows immediately.

Measurement models: Stochastic preparation operations

The case of stochastic preparation operations is now fairly easy to handle. As far as the system is concerned, we can restrict attention to the support of E. We now let d stand for the dimension of the support, and we let $|e_j\rangle$ be an arbitrary orthonormal basis for the support. Recall that the stochastic operation involves a projection Π_A onto the ancilla subspace S_A. If the system output state is to be independent of the input product state, we must have

$$|\beta_{j,l}\rangle = U|e_j\rangle \otimes |E_l\rangle = \sqrt{\mu_{j,l}} \sum_k \sqrt{\lambda_k} |f_k\rangle \otimes |F^k_{j,l}\rangle + \sqrt{1 - \mu^2_{j,l}} |\Phi_{j,l}\rangle \ , \tag{4a}$$

where the first term is in S_A and the second term is orthogonal to S_A. The sum is the Schmidt decomposition of a state that marginalizes to σ. Thus the states $|F^k_{j,l}\rangle$ are orthonormal,

$$\langle F^k_{j,l}|F^{k'}_{j,l}\rangle = \delta_{kk'} \ , \tag{4b}$$

as in Eq. (1b); moreover, these states lie in the subspace S_A. The joint states $|\Phi_{j,l}\rangle$ satisfy $\Pi_A|\Phi_{j,l}\rangle = 0$, and the real quantity $\mu_{j,l} \geq 0$ is the probability for the device to output σ when the input state is $|e_j\rangle \otimes |E_l\rangle$. It is useful to absorb the factors involving $\mu_{j,l}$ into the states:

$$|\tilde{F}^k_{j,l}\rangle = \sqrt{\mu_{j,l}}|F^k_{j,l}\rangle \qquad \text{and} \qquad |\tilde{\Phi}_{j,l}\rangle = \sqrt{1-\mu^2_{j,l}}|\Phi_{j,l}\rangle \ .$$

This puts Eq. (4a) into the form

$$|\beta_{j,l}\rangle = U|e_j\rangle \otimes |E_l\rangle = \sum_k \sqrt{\lambda_k}|f_k\rangle \otimes |\tilde{F}^k_{j,l}\rangle + |\tilde{\Phi}_{j,l}\rangle \ . \tag{4a$'$}$$

Finally, the states $|\beta_{j,l}\rangle$, numbering dD, are an orthonormal basis for the joint system, i.e.,

$$\delta_{jj'}\delta_{ll'} = \langle\beta_{j,l}|\beta_{j',l'}\rangle = \sum_k \lambda_k \langle\tilde{F}^k_{j,l}|\tilde{F}^k_{j',l'}\rangle + \langle\tilde{\Phi}_{j,l}|\tilde{\Phi}_{j',l'}\rangle \ . \tag{4c}$$

We also need to have that

$$P_A U|\psi\rangle \otimes |\Psi\rangle = \sum_{j,l} a_j b_l P_A|\beta_{j,l}\rangle = \sum_k \sqrt{\lambda_k}|f_k\rangle \otimes \left(\sum_{j,l} a_j b_l|\tilde{F}^k_{j,l}\rangle\right) \tag{5a}$$

marginalizes to a nonzero multiple of σ for all input product states $|\psi\rangle \otimes |\Psi\rangle$. This means that for any choice of the expansion coefficients a_j and b_l, the states

$$\sum_{j,l} a_j b_l|\tilde{F}^k_{j,l}\rangle \ , \quad k = 1, \ldots, \mathrm{rank}(\sigma) \ , \tag{5b}$$

must be orthogonal, and the magnitude of these states must be the same, that magnitude, $\mu_{\psi,\Psi} > 0$, being the (positive) probability to output σ for input $|\psi\rangle \otimes |\Psi\rangle$. This is a scaled-down version of orthonormality.

The joint system operators,

$$M^{kk'} = \sum_{j,l,j',l'} |e_j\rangle \otimes |E_l\rangle \underbrace{\langle\tilde{F}^k_{j,l}|\tilde{F}^{k'}_{j',l'}\rangle}_{=M^{kk'}_{j,l;j',l'}} \langle e_{j'}| \otimes \langle E_{l'}| \ ,$$

must satisfy

$$(\langle\psi| \otimes \langle\Psi|)M^{kk'}(|\psi\rangle \otimes |\Psi\rangle) = \sum_{j,l,j',l'} a^*_j a_{j'} b^*_l b_{l'} \langle\tilde{F}^k_{j,l}|\tilde{F}^{k'}_{j',l'}\rangle = \mu_{\psi,\Psi}\delta_{kk'} \ .$$

for all states $|\psi\rangle$ and $|\Psi\rangle$. This implies that $M^{kk'} = 0$ for $k \neq k'$, as in the case of deterministic preparation operations, and that the Hermitian operators M^{kk} are independent of k, i.e., $M^{kk} = M = M^\dagger$. In other words, for each k, the states $|\tilde{F}^k_{j,l}\rangle$ span a subspace

S_k, the subspaces for different values of k are orthogonal, and within each subspace, the states $|\tilde{F}^k_{j,l}\rangle$ have the same inner-product (Gram) matrix.

Now we invoke Eq. (4c):

$$\delta_{jj'}\delta_{ll'} = \sum_k \lambda_k M^{kk}_{j,l;j',l'} + \langle\tilde{\Phi}_{j,l}|\tilde{\Phi}_{j',l'}\rangle = M_{j,l;j',l'} + \langle\tilde{\Phi}_{j,l}|\tilde{\Phi}_{j',l'}\rangle \ .$$

Consider the states

$$|\gamma^k_{j,l}\rangle = |f_k\rangle \otimes |\tilde{F}^k_{j,l}\rangle + |\tilde{\Phi}_{j,l}\rangle \ ,$$

which satisfy

$$\langle\gamma^k_{j,l}|\gamma^k_{j',l'}\rangle = M_{j,l;j',l'} + \langle\tilde{\Phi}_{j,l}|\tilde{\Phi}_{j',l'}\rangle = \delta_{jj'}\delta_{ll'} \ .$$

Thus, for each k, these states are an orthonormal basis for the joint Hilbert space of dimension dD; indeed, these states are a Neumark extension of the states $|f_k\rangle \otimes |\tilde{F}^k_{j,l}\rangle$. We also have

$$\langle\gamma^k_{j,l}|(|f_{k'}\rangle \otimes |\tilde{F}^{k'}_{j,l}\rangle) = 0 \quad \text{for } k \neq k',$$

which is impossible if there is more than one value of k, i.e., if $\mathrm{rank}(\sigma) > 1$.

This leaves open the nettling case of $\mathrm{rank}(\sigma) = 1$. To handle it, we show that there is no operator M with the required properties. The proof thus applies regardless of the rank of σ [although it is overkill when $\mathrm{rank}(\sigma) > 1$] and is more akin to what we did for deterministic preparation operations. What we show is that there is a product state in the null subspace of M; for that product state $\mu_{\psi,\Psi}$ will be zero, giving a contradiction. We can drop the index k on the states $|F^k_{j,l}\rangle$, since all values of k give the same operator M.

We first note that the rank of M is the dimension of the subspace spanned by the states $|F_{j,l}\rangle$.* Since the states $|F_{j,l}\rangle$ span a space of dimension no greater than D, the rank of M is no greater than D, and the dimension of the null subspace of M is no less than $(d-1)D$. We are left with the need to show that a subspace of dimension $\geq (d-1)D$ must contain a product state. This is established by the fact the maximum dimension of a subspace that contains only entangled states is $(d-1)(D-1) < (d-1)D$ [K. R. Parthasarathy, "On the maximal dimension of a completely entangled subspace for finite level quantum systems," *Proceedings of the Indian Academy of Science (Mathematical Sciences)* **114**(4), 365–374 (2004), `arxiv:quant-ph/0405077`.

* This is a standard result: Let $M_{jk} = \langle\phi_j|\phi_k\rangle$ be a Gram matrix. Diagonalize it with a unitary matrix, i.e., $\sum_{j,k} U^*_{jl} M_{jk} U_{km} = \nu_l \delta_{lm}$. The number of nonzero eigenvalues is the rank of M_{jk}. The vectors $|\psi_l\rangle = \sum_j |\phi_j\rangle U_{jl}$ span the space spanned by the states $|\phi_j\rangle$, and since they satisfy $\langle\psi_l|\psi_m\rangle = \nu_l\delta_{lm}$, they span a space of dimension equal to the rank of M.