Completely positive maps, positive maps, and the Lindblad form
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The Lindblad form: Completely positive Markovian evolutions

Consider a one-parameter family of completely positive maps $\mathcal{C}_t = e^{\mathcal{L}t}$ that are obtained by exponentiating a superoperator generator $\mathcal{L}$. We can write a time dependent Kraus (operator-sum) decomposition

$$\mathcal{C}_t = e^{\mathcal{L}t} = \sum_\alpha B_\alpha(t) \odot B_\alpha^\dagger(t) \; . \tag{1}$$

If we require the maps to be trace-preserving, then we must have

$$1 = \mathcal{C}_t^\times(1) = \sum_\alpha B_\alpha^\dagger(t) B_\alpha(t) \; . \tag{2}$$

Notice that if $C_\epsilon = e^{\mathcal{L}\epsilon}$ is completely positive for nonzero, but arbitrarily small times ϵ, then the entire one-parameter family is completely positive.

At $t = 0$, $\mathcal{C}_t$ becomes the identity superoperator:

$$\mathcal{I} = 1 \odot 1 = \mathcal{C}_{t=0} = \sum_\alpha B_\alpha(0) \odot B_\alpha^\dagger(0) \; . \tag{3}$$

The decomposition theorem for completely positive maps then tells us that

$$B_\alpha(0) = V_{\alpha 0} 1 \; , \tag{4}$$

where the complex numbers $V_{\alpha 0}$ are the zeroth column of a unitary matrix, i.e.,

$$1 = \sum_\alpha |V_{\alpha 0}|^2 \; . \tag{5}$$

Now consider a small time $t = \epsilon$, and separate the decomposition operators $B_\alpha(\epsilon)$ into two classes: (i) The first class consists of those decomposition operators that go to a (nonzero) multiple of 1 as ϵ goes to zero, i.e., those for which $V_{\alpha 0} \neq 0$; assign these operators the indices $\alpha = 1, \ldots, N$. (ii) The second class consists of those decomposition operators that go to zero as ϵ goes to zero, i.e., those for which $V_{\alpha 0} = 0$; let these operators have the indices $\alpha > N$. Notice that the decomposition operators in the first class must have the form

$$B_\alpha(\epsilon) = V_{\alpha 0} 1 + \epsilon b_\alpha \; , \tag{6}$$

whereas those in the second class must have the form

$$B_\alpha(\epsilon) = \sqrt{\epsilon} b_\alpha \; . \tag{7}$$

Now let $V_{\alpha\beta}$ be a block-diagonal unitary matrix with the following form: in the upper left $N \times N$ block, $V_{\alpha\beta}$ is any $N \times N$ unitary that extends the zeroth column $V_{\alpha 0}$ and in the lower right block, $V_{\alpha\beta}$ is the unit matrix. Consider a new decomposition of $\mathcal{C}_\epsilon$, defined by the unitary remixing

$$A_\alpha(\epsilon) = \sum_\beta V^\dagger_{\alpha\beta} B_\beta(\epsilon) \ . \tag{8}$$

This remixing leaves the decomposition operators in the second class unchanged, but changes those in the first class to

$$A_\alpha(\epsilon) = \sum_\beta V^\dagger_{\alpha\beta}(V_{\beta 0} 1 + \epsilon b_\beta) = \delta_{\alpha 0} 1 + \epsilon a_\alpha \ , \tag{9}$$

where

$$a_\alpha = \sum_\beta V^\dagger_{\alpha\beta} b_\beta \ . \tag{10}$$

We can neglect all of the new first-class operators except $A_0(\epsilon)$. With a relabeling we have the decomposition

$$\mathcal{C}_\epsilon = e^{\mathcal{L}\epsilon} = \sum_\alpha A_\alpha(\epsilon) \odot A^\dagger_\alpha(\epsilon) \ , \tag{11}$$

where

$$A_\alpha(\epsilon) = \begin{cases} 1 + \epsilon a_0 = 1 - \epsilon(g/2 + ih) \ , & \alpha = 0. \\ \sqrt{\epsilon} a_\alpha \ , & \alpha > 0. \end{cases} \tag{12}$$

Here g and h are Hermitian.

Now we're set. The operation C_ϵ becomes

$$\begin{aligned} C_\epsilon = e^{\mathcal{L}\epsilon} &= \mathcal{I} + \epsilon\mathcal{L} \\ &= A_0(\epsilon) \odot A^\dagger_0(\epsilon) + \sum_\alpha A_\alpha(\epsilon) \odot A^\dagger_\alpha(\epsilon) \\ &= (1 + \epsilon a_0) \odot (1 + \epsilon a^\dagger_0) + \epsilon \sum_\alpha a_\alpha \odot a^\dagger_\alpha \\ &= \mathcal{I} + \epsilon\left(a_0 \odot 1 + 1 \odot a^\dagger_0 + \sum_\alpha a_\alpha \odot a^\dagger_\alpha \right) \ , \end{aligned} \tag{13}$$

where here and henceforth, sums run over $\alpha > 0$. Thus the generator $\mathcal{L}$ has the form

$$\begin{aligned} \mathcal{L} &= a_0 \odot 1 + 1 \odot a^\dagger_0 + \sum_\alpha a_\alpha \odot a^\dagger_\alpha \\ &= -i(h \odot 1 - 1 \odot h) - \frac{1}{2}(g \odot 1 + 1 \odot g) + \sum_\alpha a_\alpha \odot a^\dagger_\alpha \ . \end{aligned} \tag{14}$$

The trace-preserving condition, should we wish to require it, is

$$1 = \mathcal{C}^\times_\epsilon(1) = 1 + \epsilon\mathcal{L}^\times(1) \ , \tag{15}$$

which becomes

$$0=\mathcal{L}^\times(1)=-g+\sum_\alpha a_\alpha^\dagger a_\alpha \quad\Longleftrightarrow\quad g=\sum_\alpha a_\alpha^\dagger a_\alpha\ . \tag{16}$$

In the trace-preserving case, the first term in $\mathcal{L}$ is a Hamiltonian evolution, and the remaining terms describe a nonunitary evolution consistent with complete positivity.

It is useful to write the generator in a more abstract form. The final term in $\mathcal{L}$ is a completely positive "diffusion" map

$$\mathcal{D}=\sum_\alpha a_\alpha\odot a_\alpha^\dagger=\sum_\alpha |\,a_\alpha)(a_\alpha|\ . \tag{17}$$

We can write the generator (14) in the abstract form

$$\mathcal{L}=-i(h\odot 1-1\odot h)-(g\odot 1+1\odot g)/2+\mathcal{D}\ . \tag{18}$$

The trace-preserving condition is

$$0=\mathcal{L}^\times(1)=-g+\mathcal{D}^\times(1) \quad\Longleftrightarrow\quad g=\mathcal{D}^\times(1)=\sum_\alpha a_\alpha^\dagger a_\alpha\ , \tag{19}$$

since

$$\mathcal{L}^\times=i(h\odot 1-1\odot h)-(g\odot 1+1\odot g)/2+\mathcal{D}^\times\ . \tag{20}$$

This leaves the trace-preserving generator in the form

$$\mathcal{L}=-i(h\odot 1-1\odot h)+\frac{1}{2}\sum_\alpha\Big(2a_\alpha\odot a_\alpha^\dagger-1\odot a_\alpha^\dagger a_\alpha-a_\alpha^\dagger a_\alpha\odot 1\Big)\ . \tag{21}$$

What we have shown is that any generator of a one-parameter family of completely positive maps has the form (18), with $\mathcal{D}$ completely positive and with g given by (19) if the maps are trace-preserving. Moreover, it is clear from our construction that any generator of the form (18) does generate a one-parameter family of completely positive maps, which are trace-preserving if g is given by (19). On the other hand, not all completely positive maps can be derived from Markovian evolutions, as shown by Wolf and Cirac [M. M. Wolf and J. I. Cirac, "Dividing quantum channels," *Communications in Mathematical Physics* **279**, 147–168 (2008)].

The freedom in defining a particular generator $\mathcal{L}$, if one wishes the diffusion map to be given in terms of a Kraus decomposition, is the freedom to do arbitrary unitary remixings of the decomposition operators a_α. Among the Kraus decompositions of $\mathcal{D}$, there is a special one, the eigendecomposition of $\mathcal{D}$ relative to the left-right action:

$$\mathcal{D}=\sum_\alpha \tau_\alpha\odot\tau_\alpha^\dagger=\sum_\alpha|\tau_\alpha)(\tau_\alpha|\ ,\quad\text{where}\quad(\tau_\alpha|\tau_\beta)=\mathrm{tr}(\tau_\alpha^\dagger\tau_\beta)=\lambda_\alpha\delta_{\alpha\beta}\ ,\quad \lambda_\alpha>0\ . \tag{22}$$

The diffusion map is generally not given in terms of a Kraus decomposition, however. The general form of the diffusion map is obtained by defining new operators

$$c_\beta = \sum_\alpha \tau_\alpha L^{-1}_{\alpha\beta} = \sum_\alpha \frac{\tau_\alpha}{\sqrt{\lambda_\alpha}} \sqrt{\lambda_\alpha} L^{-1}_{\alpha\beta} \qquad \Longleftrightarrow \qquad \tau_\alpha = \sum_\beta c_\beta L_{\beta\alpha} = \sqrt{\lambda_\alpha} \sum_\beta c_\beta \frac{L_{\beta\alpha}}{\sqrt{\lambda_\alpha}} \ , \tag{23}$$

where L is an invertible matrix (we pad the list of eigenoperators with zero operators if there are, as is generally the case, more operators c_β than eigenoperators τ_α). The operators c_β provide a Kraus decomposition of $\mathcal{D}$ if and only if L is unitary. Generally the operators c_β are an overcomplete and/or nonorthogonal set; they are a complete, orthonormal set of operators if and only if the matrix $L_{\beta\alpha}/\sqrt{\lambda_\alpha}$ is unitary. The diffusion superoperator now assumes the form

$$\mathcal{D} = \sum_{\alpha,\beta} A_{\alpha\beta} c_\alpha \odot c_\beta^\dagger = \sum_{\alpha,\beta} A_{\alpha\beta} |\, c_\alpha)(c_\beta| \ , \tag{24}$$

where

$$A_{\alpha\beta} = \sum_\gamma L_{\alpha\gamma} L^*_{\beta\gamma} = (LL^\dagger)_{\alpha\beta} \tag{25}$$

is a positive matrix (this expresses the complete positivity of $\mathcal{D}$). Notice that for trace-preserving situations, where

$$g = \mathcal{D}^\times(1) = \sum_{\alpha,\beta} A^*_{\alpha\beta} c_\alpha^\dagger c_\beta = \sum_{\alpha,\beta} A_{\alpha\beta} c_\beta^\dagger c_\alpha \ , \tag{26}$$

the generator takes the form

$$\mathcal{L} = -i(h \odot 1 - 1 \odot h) + \frac{1}{2} \sum_{\alpha,\beta} A_{\alpha\beta} \Big(2 c_\alpha \odot c_\beta^\dagger - 1 \odot c_\beta^\dagger c_\alpha - c_\beta^\dagger c_\alpha \odot 1 \Big) \ . \tag{27}$$

This general form for trace-preserving $\mathcal{L}$ is called the *Lindblad form* [R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 1987)].

Positive and left-right Hermitian Markovian evolutions

What changes if the one-parameter family $\mathcal{C}_t$ is supposed to consist of positive, but not necessarily completely positive maps? More generally, we should ask about generators $\mathcal{L}$ of one-parameter families $\mathcal{C}_t = e^{\mathcal{L}t}$ of left-right Hermitian maps, of which positive maps are a special case. A map is left-right Hermitian if and only if it is the difference between two completely positive maps (equivalently, a map is left-right Hermitian if and only if it maps Hermitian operators to Hermitian operators). Notice that if $\mathcal{C}_\epsilon$ is left-right Hermitian (positive) for nonzero, but arbitrarily small times ϵ, then the entire one-parameter family is left-right Hermitian (positive).

For a small time ϵ, we can write

$$\mathcal{C}_\epsilon = C_{1,\epsilon} - C_{2,\epsilon} \ , \tag{28}$$

where $C_{1,\epsilon}$ and $C_{2,\epsilon}$ are completely positive. As ϵ goes to zero, C_1 must go to the unit superoperator, and C_2 must go zero. We can conclude immediately that $C_{2,\epsilon}$ has the form

$$C_{2,\epsilon} = \epsilon\mathcal{L}_2 = \epsilon\mathcal{D}_2 \ , \tag{29}$$

with $\mathcal{D}_2$ completely positive, and we can apply the argument above to $C_{1,\epsilon}$ to get

$$C_{1,\epsilon} = \mathcal{I} + \epsilon\mathcal{L}_1 \ , \tag{30}$$

where

$$\mathcal{L}_1 = -i(h \odot 1 - 1 \odot h) - (g \odot 1 + 1 \odot g)/2 + \mathcal{D}_1 \ , \tag{31}$$

with $\mathcal{D}_1$ completely positive. Thus the generator of $\mathcal{C}_\epsilon$ is

$$\mathcal{L} = \mathcal{L}_1 - \mathcal{L}_2 = -i(h \odot 1 - 1 \odot h) - (g \odot 1 + 1 \odot g)/2 + \mathcal{D} \ , \tag{32}$$

where

$$\mathcal{D} = \mathcal{D}_1 - \mathcal{D}_2 \tag{33}$$

is left-right Hermitian. If we want the maps to be trace-preserving, we again have the condition

$$0 = \mathcal{L}^\times(1) = -g + \mathcal{D}^\times(1) \quad \Longleftrightarrow \quad g = \mathcal{D}^\times(1) = \frac{1}{2}[\mathcal{D}_1^\times(1) - \mathcal{D}_2^\times(1)] \ . \tag{34}$$

What we have shown is that the generator of a one-parameter family of left-right Hermitian maps has the form (32). Moreover, the construction shows that any generator of this form does generate a one-parameter family of left-right Hermitian maps. [This also follows directly from two facts: (i) $(\mathcal{A} \circ \mathcal{B})^\dagger = \mathcal{A}^\dagger \circ \mathcal{B}^\dagger$, where composition, denoted by $\circ$, is the kind of multiplication in $e^{\mathcal{L}t}$, and (ii)

$$\begin{aligned} \mathcal{L}^\dagger &= i(1 \odot h - h \odot 1) - (1 \odot + - g \odot 1)/2 + \mathcal{D}^\dagger \\ &= -i(h \odot 1 - 1 \odot h) - (g \odot 1 + 1 \odot g)/2 + \mathcal{D} = \mathcal{L} \ . \end{aligned} \tag{35}$$

These two facts imply that $\mathcal{C}_t^\dagger = e^{\mathcal{L}^\dagger t} = e^{\mathcal{L}t} = \mathcal{C}_t$.] We do not address the question of whether all left-right Hermitian maps can be generated by this procedure, although it seems very unlikely to me that they can be.

Chris Fuchs has suggested that positive maps cannot be generated from the identity as part of a one-parameter family, but it is clear that some positive maps can be so generated, so this property does not distinguish completely positive maps from positive ones.