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Subject: **Simple proof that a superoperator that maps Hermitians to Hermitians is left-right Hermitian**
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Notice that
$$\begin{aligned}\big\langle e_l\big|\mathcal{A}^\dagger(|e_j\rangle\langle e_k|\,)\big|e_m\big\rangle &= \big(|e_l\rangle\langle e_j|\,\big|\mathcal{A}^\dagger\big||e_m\rangle\langle e_k|\,\big)\\ &= \big(|e_m\rangle\langle e_k|\,\big|\mathcal{A}\big||e_l\rangle\langle e_j|\,\big)^*\\ &= \big\langle e_m\big|\mathcal{A}(|e_k\rangle\langle e_j|\,)\big|e_l\big\rangle^*\\ &= \big\langle e_l\big|[\mathcal{A}(|e_k\rangle\langle e_j|\,)]^\dagger\big|e_m\big\rangle\ ,\end{aligned}$$
which implies that
$$\mathcal{A}^\dagger(|e_j\rangle\langle e_k|\,) = [\mathcal{A}(|e_k\rangle\langle e_j|\,)]^\dagger\ .$$
Thus we have that
$$\mathcal{A}=\mathcal{A}^\dagger \quad\Longleftrightarrow\quad \mathcal{A}(|e_j\rangle\langle e_k|\,) = [\mathcal{A}(|e_k\rangle\langle e_j|\,)]^\dagger\ .$$

Theorem. $\mathcal{A}(H)=\mathcal{A}(H)^\dagger$ for all $H=H^\dagger \Longleftrightarrow \mathcal{A}=\mathcal{A}^\dagger$

$\Longleftarrow$: Write H in terms of its eigendecomposition:
$$H=\sum_j \lambda_j|e_j\rangle\langle e_j|\ .$$
Then
$$\mathcal{A}(H)=\sum_j\lambda_j\mathcal{A}\big(|e_j\rangle\langle e_j|\big)=\sum_j\lambda_j[\mathcal{A}\big(|e_j\rangle\langle e_j|\big)]^\dagger=\mathcal{A}(H)^\dagger\ .$$
Equivalently, since $\mathcal{A}=\mathcal{A}^\dagger$, one can write $\mathcal{A}$ in terms of its eigendecomposition:
$$\mathcal{A}=\sum_\alpha\lambda_\alpha|\tau_\alpha)(\tau_\alpha|=\sum_\alpha\lambda_\alpha\tau_\alpha\odot\tau_\alpha^\dagger\ .$$
Then
$$\mathcal{A}(H)=\sum_\alpha\lambda_\alpha\tau_\alpha H\tau_\alpha^\dagger=\mathcal{A}(H)^\dagger\ .$$

$\Longrightarrow$:

$$\begin{aligned}
\mathcal{A}(|e_j\rangle\langle e_k|\,) &= \mathcal{A}\left(\frac{1}{2}\big(|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|\,\big) + i\frac{-i}{2}\big(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\,\big)\right) \\
&= \mathcal{A}\left(\frac{1}{2}\big(|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|\,\big)\right) + i\mathcal{A}\left(\frac{-i}{2}\big(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\,\big)\right) \\
&= \left[\mathcal{A}\left(\frac{1}{2}\big(|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|\,\big)\right)\right]^\dagger + i\left[\mathcal{A}\left(\frac{-i}{2}\big(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\,\big)\right)\right]^\dagger \\
&= \left[\mathcal{A}\left(\frac{1}{2}\big(|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|\,\big)\right) - i\mathcal{A}\left(\frac{-i}{2}\big(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\,\big)\right)\right]^\dagger \\
&= \left[\mathcal{A}\left(\frac{1}{2}\big(|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|\,\big) - i\frac{-i}{2}\big(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\,\big)\right)\right]^\dagger \\
&= [\mathcal{A}(|e_k\rangle\langle e_j|\,)]^\dagger\ .
\end{aligned}$$

Therefore, $\mathcal{A} = \mathcal{A}^\dagger$.