Entropy of Hilbert-space splits into two equal parts

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Let $|e_j\rangle$, $j = 1, \ldots, D$, be an orthonormal basis for a D-dimensional Hilbert space. Let

$$P_+ = \sum_{j=1}^{n} |e_j\rangle\langle e_j| \tag{1}$$

be the projector onto the subspace S_+ spanned by the first n vectors,

$$P_- = \sum_{j=n+1}^{n+m} |e_j\rangle\langle e_j| \tag{2}$$

be the projector onto the subspace S_- spanned by the next m vectors, and $P_0 = I - P_+ - P_-$ be the projector onto the subspace S_0 spanned by the remaining $D - n - m$ vectors. An arbitrary normalized vector can be expanded uniquely as

$$|\psi\rangle = \cos\xi\big(\cos\theta|\chi\rangle + \sin\theta|\eta\rangle\big) + \sin\xi|\phi\rangle\;, \tag{3}$$

where $|\chi\rangle \in S_+$, $|\eta\rangle \in S_-$, and $|\phi\rangle \in S_0$ are normalized vectors. The angle ξ characterizes a split of projective Hilbert space between subspace S_0 and the span of S_+ and S_-, and the angle θ characterizes a further split between S_+ and S_-.

We are interested in the density operator formed from all pure states closer to the subspace S_+ than an angle Θ,

$$\rho = \mathcal{N}\int_{\theta\le\Theta} d\mathcal{S}_{2D-1}\,|\psi\rangle\langle\psi|\;, \tag{4}$$

where $d\mathcal{S}_{2D-1}$ is the integration measure on the $(2D-1)$-sphere and $\mathcal{N}$ is a normalization factor. It is easy to see that this density operator has the form

$$\rho = \lambda_+ P_+ + \lambda_- P_- + \lambda_0 P_0\;. \tag{5}$$

Our job is to determine the three eigenvalues, $\lambda_\pm$ and λ_0, which satisfy

$$n\lambda_+ + m\lambda_- + (D - n - m)\lambda_0 = 1\;. \tag{6}$$

It turns out that $\lambda_0 = 1/D$, so we have

$$\lambda_- = \frac{1}{m}\left(1 - \frac{D-n-m}{D} - n\lambda_+\right) = \frac{1}{D}\left(1 + \frac{n}{m}(1 - D\lambda_+)\right)\;. \tag{7}$$

A small change in $|\psi\rangle$ can be written as

$$
\begin{aligned}
|d\psi\rangle = d\xi\big(-\sin\xi(\cos\theta|\chi\rangle + \sin\theta|\eta\rangle) + \cos\xi|\phi\rangle\big) + \sin\xi|d\phi\rangle \\
+ \cos\xi\big(d\theta(-\sin\theta|\chi\rangle + \cos\theta|\eta\rangle) + \cos\theta|d\chi\rangle + \sin\theta|d\eta\rangle\big) \ .
\end{aligned}
\tag{8}
$$

This gives a line element on normalized vectors,

$$
ds^2 = \langle d\psi|d\psi\rangle = d\xi^2 + \sin^2\!\xi\langle d\phi|d\phi\rangle + \cos^2\!\xi\big(d\theta^2 + \cos^2\!\theta\langle d\chi|d\chi\rangle + \sin^2\!\theta\langle d\eta|d\eta\rangle\big) \ , \tag{9}
$$

and a corresponding volume element on the $(2D-1)$-sphere of normalized vectors

$$
\begin{aligned}
d\mathcal{S}_{2D-1} = {} & \sin^{2(D-n-m)-1}\xi\,\cos^{2(n+m)-1}\xi\,d\xi \\
& \times \cos^{2n-1}\theta\,\sin^{2m-1}\theta\,d\theta\,d\mathcal{S}_{2(D-n-m)-1}\,d\mathcal{S}_{2n-1}\,d\mathcal{S}_{2m-1} \ .
\end{aligned}
\tag{10}
$$

Normalizing the density operator gives

$$
\begin{aligned}
1 = \mathrm{tr}(\rho) &= \mathcal{N}\int_{\theta\le\Theta} d\mathcal{S}_{2D-1} \\
&= \mathcal{N}\mathcal{S}_{2(D-n-m)-1}\mathcal{S}_{2n-1}\mathcal{S}_{2m-1}\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)-1}\xi\,\cos^{2(n+m)-1}\xi \\
&\qquad \times \int_0^{\Theta} d\theta\ \cos^{2n-1}\theta\,\sin^{2m-1}\theta \ .
\end{aligned}
\tag{11}
$$

Let's first verify that $\lambda_0 = 1/D$. Letting $|e_0\rangle$ be any normalized vector in S_0, we have

$$
\begin{aligned}
\lambda_0 &= \langle e_0|\rho|e_0\rangle \\
&= \mathcal{N}\int_{\theta\le\Theta} d\mathcal{S}_{2D-1}\ \underbrace{|\langle e_0|\psi\rangle|^2}_{=\sin^2\xi|\langle e_0|\phi\rangle|^2} \\
&= \mathcal{N}\mathcal{S}_{2n-1}\mathcal{S}_{2m-1}\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)+1}\xi\,\cos^{2(n+m)-1}\xi \\
&\qquad \times \int_0^{\Theta} d\theta\ \cos^{2n-1}\theta\,\sin^{2m-1}\theta\ \underbrace{\int d\mathcal{S}_{2(D-n-m)-1}\,|\langle e_0|\phi\rangle|^2}_{=\dfrac{\mathcal{S}_{2(D-n-m)-1}}{D-n-m}} \ .
\end{aligned}
\tag{12}
$$

Plugging in the normalization constant from Eq. (11) gives

$$
\begin{aligned}
\lambda_0 &= \frac{1}{D-n-m}\,\frac{\displaystyle\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)+1}\xi\cos^{2(n+m)-1}\xi}{\displaystyle\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)-1}\xi\cos^{2(n+m)-1}\xi} \\
&= \frac{1}{D-n-m}\,\frac{\displaystyle\int_0^1 du\, u^{D-n-m}(1-u)^{n+m-1}}{\displaystyle\int_0^1 du\, u^{D-n-m-1}(1-u)^{n+m-1}} \\
&= \frac{1}{D-n-m}\,\frac{\Gamma(D-n-m+1)\Gamma(n+m)/\Gamma(D+1)}{\Gamma(D-n-m)\Gamma(n+m)/\Gamma(D)} \\
&= \frac{1}{D}\,.
\end{aligned}
\tag{13}
$$

What about λ_+? Letting $|e_+\rangle$ be any normalized vector in S_+, we have

$$
\begin{aligned}
\lambda_+ &= \langle e_+|\rho|e_+\rangle \\
&= \mathcal{N}\int_{\theta\le\Theta} d\mathcal{S}_{2D-1}\qquad \underbrace{|\langle e_+|\psi\rangle|^2}_{=\cos^2\xi\cos^2\theta|\langle e_+|\chi\rangle|^2} \\
&= \mathcal{N}\mathcal{S}_{2(D-n-m)-1}\mathcal{S}_{2m-1}\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)-1}\xi\cos^{2(n+m)+1}\xi \\
&\quad\times\int_0^{\Theta} d\theta\ \cos^{2n+1}\theta\sin^{2m-1}\theta\ \underbrace{\int d\mathcal{S}_{2n-1}\,|\langle e_+|\chi\rangle|^2}_{=\frac{\mathcal{S}_{2n-1}}{n}}\ .
\end{aligned}
\tag{14}
$$

Again plugging in the normalization constant, we get

$$
\begin{aligned}
\lambda_+ &= \frac{1}{n}\,\frac{\displaystyle\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)-1}\xi\cos^{2(n+m)+1}\xi}{\displaystyle\int_0^{\pi/2} d\xi\ \sin^{2(D-n-m)-1}\xi\cos^{2(n+m)-1}\xi}\,\frac{\displaystyle\int_0^{\Theta} d\theta\ \cos^{2n+1}\theta\sin^{2m-1}\theta}{\displaystyle\int_0^{\Theta} d\theta\ \cos^{2n-1}\theta\sin^{2m-1}\theta} \\
&= \frac{1}{n}\qquad \underbrace{\frac{\displaystyle\int_0^1 du\, u^{D-n-m-1}(1-u)^{n+m}}{\displaystyle\int_0^1 du\, u^{D-n-m-1}(1-u)^{n+m-1}}}_{\displaystyle =\frac{\Gamma(D-n-m)\Gamma(n+m+1)/\Gamma(D+1)}{\Gamma(D-n-m)\Gamma(n+m)/\Gamma(D)} =\frac{n+m}{D}}\qquad \frac{\displaystyle\int_0^{\sin^2\Theta} dv\, v^{m-1}(1-v)^n}{\displaystyle\int_0^{\sin^2\Theta} dv\, v^{m-1}(1-v)^{n-1}}\,,
\end{aligned}
\tag{15}
$$

and this gives

$$\lambda_+ = \frac{n+m}{nD} \frac{\displaystyle\int_0^{\sin^2\Theta} dv\, v^{m-1}(1-v)^n}{\displaystyle\int_0^{\sin^2\Theta} dv\, v^{m-1}(1-v)^{n-1}} . \tag{16}$$

We now specialize to the case of interest, $n = m$ and $\Theta = \pi/4$, so that ρ is constructed from pure states occupying one of two halves of Hilbert space:

$$\lambda_+ = \frac{2}{D} \frac{\displaystyle\int_0^{1/2} dv\, v^{n-1}(1-v)^n}{\displaystyle\int_0^{1/2} dv\, v^{n-1}(1-v)^{n-1}} . \tag{17}$$

Define the integral

$$I(n,m) = 2^{n+m} \int_0^{1/2} dv\,(1-v)^n v^{m-1} . \tag{18}$$

For $m \geq 1$, integrating by parts gives

$$\begin{aligned} I(n,m) &= 2^{n+m} \left((1-v)^n \frac{v^m}{m}\bigg|_0^{1/2} + \frac{n}{m} \int_0^{1/2} dv\,(1-v)^{n-1} v^m \right) \\ &= \frac{1}{m} + \frac{n}{m} I(n-1, m+1) , \end{aligned} \tag{19}$$

which when combined with

$$I(0,m) = 2^m \int_0^{1/2} dv\, v^{m-1} = 2^m \left. \frac{v^m}{m} \right|_0^{1/2} = 1/m , \tag{20}$$

allows us to find $I(n,m)$ recursively,

$$I(n,m) = n!(m-1)! \sum_{k=0}^{n} \frac{1}{(n-k)!(m+k)!} . \tag{21}$$

This allows us to write

$$\begin{aligned} \int_0^{1/2} dv\, v^{n-1}(1-v)^n &= \frac{I(n,n)}{2^{2n}} = \frac{n!(n-1)!}{2^{2n}} \sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!} , \\ \int_0^{1/2} dv\, v^{n-1}(1-v)^{n-1} &= \frac{I(n-1,n)}{2^{2n-1}} = \frac{[(n-1)!]^2}{2^{2n-1}} \sum_{k=0}^{n-1} \frac{1}{(n-k-1)!(n+k)!} . \end{aligned} \tag{22}$$

Mathematica does the sums as

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!} &= \frac{2^{2n-1}}{(2n)!}\Bigg(1+\underbrace{\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,n!}}_{\substack{=\frac{(2n-1)!!}{2^n n!} \\ =\frac{(2n)!}{2^{2n}(n!)^2}}}\Bigg)\,, \\
\sum_{k=0}^{n-1} \frac{1}{(n-k-1)!(n+k)!} &= \frac{2^{2(n-1)}}{(2n-1)!}\,,
\end{aligned}
\tag{23}
$$

which gives us

$$
\begin{aligned}
\int_0^{1/2} dv\, v^{n-1}(1-v)^n &= \frac{n!(n-1)!}{2(2n)!}\left(1+\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,n!}\right)\,, \\
\int_0^{1/2} dv\, v^{n-1}(1-v)^{n-1} &= \frac{[(n-1)!]^2}{2(2n-1)!}\,.
\end{aligned}
\tag{24}
$$

Plugging these results into Eqs. (17) and (7), we get

$$
\lambda_\pm = \frac{1}{D}\left(1\pm\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,n!}\right) = \frac{1}{D}\left(1\pm\frac{(2n-1)!!}{2^n n!}\right) = \frac{1}{D}\left(1\pm\frac{(2n)!}{2^{2n}(n!)^2}\right)\,. \tag{25}
$$

When $n=1$, we get $\lambda_+ = 3/2D$ and $\lambda_- = 1/2D$, and when $n=2$, $\lambda_+ = 11/8D$ and $\lambda_- = 5/8D$. For large n (and D), we can write

$$
\frac{\Gamma(n+1/2)}{n!} = \frac{1}{\sqrt{n}} + o\!\left(\frac{1}{n^{3/2}}\right)\,, \tag{26}
$$

which gives

$$
\lambda_\pm = \frac{1}{D}\left(1\pm\frac{1}{\sqrt{\pi n}} + o\!\left(\frac{1}{n^{3/2}}\right)\right)\,. \tag{27}
$$

The entropy can be put in the form

$$
\begin{aligned}
S(\rho) &= -n\lambda_+\log\lambda_+ - n\lambda_-\log\lambda_- - (D-2n)\lambda_0\log\lambda_0 \\
&= -\frac{2n}{D}\Bigg(\frac{D\lambda_+}{2}\Big(\log(D\lambda_+/2)-\log(D/2)\Big) + \frac{D\lambda_-}{2}\Big(\log(D\lambda_-/2)-\log(D/2)\Big) \\
&\qquad + \left(1-\frac{2n}{D}\right)\log D\Bigg) \\
&= \log D - \frac{2n}{D}\big(1-H_2(D\lambda_+/2)\big)\,.
\end{aligned}
\tag{28}
$$

For fixed D, this is a decreasing function of n. For large D, the entropy decreases to $\log D - 1/\pi D \ln 2$ as n becomes large. We get at this behavior by noting that for large n, we have

$$\frac{D\lambda_\pm}{2} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{\pi n}} + o\left(\frac{1}{n^{3/2}}\right)\right) , \tag{29}$$

which gives the following approximate forms for the binary entropy,

$$H_2(D\lambda_+/2) = 1 - \frac{1}{2\ln 2}\left(\frac{(D\lambda_+/2 - 1/2)^2}{1/2} + \frac{(D\lambda_-/2 - 1/2)^2}{1/2}\right) = 1 - \frac{1}{2\pi n \ln 2} , \tag{30}$$

and the entropy,

$$S(\rho) = \log D - \frac{1}{\pi D \ln 2} = \log D - \frac{0.459}{D} . \tag{31}$$