

Operator formalism and quasidistributions for creation and annihilation operators

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This document is essentially a list of equations. It contains the following sections, each of which begins a new page. Pages are numbered individually by section. Equations are numbered sequentially through the whole document. The document is updated periodically as corrections are made and new results are included.

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In the following, A&S refers to M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (U.S. Government Printing Office, Washington, D.C., 1964). The numbers after A&S direct you to particular expressions in Abramowitz and Stegun. As far as I can tell, Wolfram MathWorld tends to use the notation of Abramowitz and Stegun, and Wikipedia pages tend to use that notation as well.

1. Basics

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x + ip) & x &= \frac{1}{\sqrt{2}}(a + a^\dagger) & [a, a^\dagger] &= 1 \\ a^\dagger &= \frac{1}{\sqrt{2}}(x - ip) & p &= -\frac{i}{\sqrt{2}}(a - a^\dagger) & [x, p] &= i \end{aligned} \quad (1)$$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{1}{4} |\langle [x, p] \rangle|^2 = \frac{1}{4} \quad (2)$$

$$\begin{aligned} a^\dagger a &= \frac{1}{2}(x^2 + p^2 - 1) & a^2 &= \frac{1}{2}(x^2 - p^2 + i(xp + px)) \\ aa^\dagger &= \frac{1}{2}(x^2 + p^2 + 1) & (a^\dagger)^2 &= \frac{1}{2}(x^2 - p^2 - i(xp + px)) \end{aligned} \quad (3)$$

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \quad [a^\dagger, a^n] = -na^{n-1} \quad (4)$$

$$[x, p^n] = inp^{n-1} \quad [p, x^n] = -inx^{n-1} \quad (5)$$

(Use $[A, B^n] = nB^{n-1}[A, B]$ if B commutes with $[A, B]$.)

$$[a, a^\dagger a] = [a, a^\dagger]a = a \quad (6)$$

$$\begin{aligned} e^{i\theta a^\dagger a} a e^{-i\theta a^\dagger a} &= a e^{-i\theta} & e^{\lambda a^\dagger a} a e^{-\lambda a^\dagger a} &= a e^{-\lambda} \\ e^{i\theta a^\dagger a} a^\dagger e^{-i\theta a^\dagger a} &= a^\dagger e^{i\theta} & e^{\lambda a^\dagger a} a^\dagger e^{-\lambda a^\dagger a} &= a^\dagger e^{\lambda} \end{aligned} \quad (7)$$

(Use $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots$)

$$\begin{aligned} e^{i\theta a^\dagger a} x e^{-i\theta a^\dagger a} &= x \cos \theta + p \sin \theta \\ e^{i\theta a^\dagger a} p e^{-i\theta a^\dagger a} &= -x \sin \theta + p \cos \theta \end{aligned} \quad (8)$$

2. Position and momentum bases

$$\begin{aligned} \langle x'|x\rangle &= \delta(x-x') & \langle p'|p\rangle &= \delta(p-p') & \langle x|p\rangle &= \frac{1}{\sqrt{2\pi}} e^{ipx} \\ \int dx |x\rangle\langle x| &= 1 & \int dp |p\rangle\langle p| &= 1 \end{aligned} \quad (9)$$

$$\begin{aligned} e^{-ipa}|x\rangle &= |x+a\rangle & \langle x|e^{-ipa} &= \langle x-a| \\ e^{ixb}|p\rangle &= |p+b\rangle & \langle p|e^{ixb} &= \langle p-b| \end{aligned} \quad (10)$$

$$\langle x|e^{-ipa}|\psi\rangle = \langle x-a|\psi\rangle \quad \langle p|e^{ixb}|\psi\rangle = \langle p-b|\psi\rangle \quad (11)$$

$$\begin{aligned} \langle x|p|\psi\rangle &= \frac{1}{i} \frac{d}{da} \langle x|e^{ipa}|\psi\rangle \Big|_{a=0} = \frac{1}{i} \frac{d}{da} \langle x+a|\psi\rangle \Big|_{a=0} = \frac{1}{i} \frac{d}{dx} \langle x|\psi\rangle \quad \Longleftrightarrow \quad p \leftrightarrow \frac{1}{i} \frac{d}{dx} \\ \langle p|x|\psi\rangle &= i \frac{d}{db} \langle p|e^{-ixb}|\psi\rangle \Big|_{b=0} = i \frac{d}{db} \langle p+b|\psi\rangle \Big|_{b=0} = i \frac{d}{dp} \langle p|\psi\rangle \quad \Longleftrightarrow \quad x \leftrightarrow i \frac{d}{dp} \end{aligned} \quad (12)$$

$$a = \frac{1}{\sqrt{2}}(x + ip) \leftrightarrow \begin{cases} \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right) & \text{(position basis)} \\ \frac{i}{\sqrt{2}} \left(\frac{d}{dp} + p \right) & \text{(momentum basis)} \end{cases} \quad (13)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \leftrightarrow \begin{cases} \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right) & \text{(position basis)} \\ \frac{i}{\sqrt{2}} \left(\frac{d}{dp} - p \right) & \text{(momentum basis)} \end{cases} \quad (14)$$

3. Displacement operator

$$D(a, \alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} = e^{i(\alpha_2 x - \alpha_1 p)} = e^{-i\alpha_1 \alpha_2 / 2} e^{i\alpha_2 x} e^{-i\alpha_1 p} = e^{i\alpha_1 \alpha_2 / 2} e^{-i\alpha_1 p} e^{i\alpha_2 x} \quad (15)$$

$$\alpha = \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) \quad \begin{aligned} \alpha_1 &= \frac{1}{\sqrt{2}}(\alpha + \alpha^*) \\ \alpha_2 &= -\frac{i}{\sqrt{2}}(\alpha - \alpha^*) \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial}{\partial \alpha} &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \alpha_1} - i \frac{\partial}{\partial \alpha_2} \right) & \frac{\partial}{\partial \alpha_1} &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*} \right) \\ \frac{\partial}{\partial \alpha^*} &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \alpha_1} + i \frac{\partial}{\partial \alpha_2} \right) & \frac{\partial}{\partial \alpha_2} &= \frac{i}{\sqrt{2}} \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha^*} \right) \end{aligned} \quad (17)$$

$$\frac{\partial^2}{\partial \alpha \partial \alpha^*} = \frac{1}{2} \left(\frac{\partial^2}{\partial \alpha_1^2} + \frac{\partial^2}{\partial \alpha_2^2} \right) \quad (18)$$

$$D^{-1}(a, \alpha) = D^\dagger(a, \alpha) = D(a, -\alpha) = D(-a, \alpha) \quad (19)$$

$$\begin{aligned} D(a, \alpha)|x\rangle &= e^{i\alpha_1 \alpha_2 / 2} e^{i\alpha_2 x} |x + \alpha_1\rangle & \langle x|D(a, \alpha) &= e^{-i\alpha_1 \alpha_2 / 2} e^{i\alpha_2 x} \langle x - \alpha_1| \\ D(a, \alpha)|p\rangle &= e^{-i\alpha_1 \alpha_2 / 2} e^{-i\alpha_1 p} |p + \alpha_2\rangle & \langle p|D(a, \alpha) &= e^{i\alpha_1 \alpha_2 / 2} e^{-i\alpha_1 p} \langle p - \alpha_2| \end{aligned} \quad (20)$$

$$\begin{aligned} \langle x|D(a, \alpha)|x'\rangle &= e^{-i\alpha_1 \alpha_2 / 2} e^{i\alpha_2 x} \delta(x - x' - \alpha_1) \\ \langle p|D(a, \alpha)|p'\rangle &= e^{i\alpha_1 \alpha_2 / 2} e^{-i\alpha_1 p} \delta(p - p' - \alpha_2) \end{aligned} \quad (21)$$

$$D(\alpha, \beta) = e^{\beta \alpha^* - \beta^* \alpha} = e^{2i(\beta_I \alpha_R - \beta_R \alpha_I)} = e^{i(\beta_2 \alpha_1 - \beta_1 \alpha_2)} \quad (22)$$

$$D(\alpha, \alpha) = 1 \quad D(\alpha, r\alpha) = 1, \quad r \text{ real} \quad (23)$$

$$D^*(\alpha, \beta) = D(\alpha^*, \beta^*) = D(\alpha, -\beta) = D(-\alpha, \beta) = D(\beta, \alpha) \quad (24)$$

$$D(a, \alpha) = e^{-|\alpha|^2 / 2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{|\alpha|^2 / 2} e^{-\alpha^* a} e^{\alpha a^\dagger} \quad (25)$$

(Use BCH: $e^{A+B} = e^{-[A,B]/2} e^A e^B$ if A and B commute with $[A, B]$.)

$$D^\dagger(a, \alpha) a D(a, \alpha) = a + \alpha \quad (26)$$

$$D^\dagger(a, \alpha) D(a, \beta) D(a, \alpha) = D(a + \alpha, \beta) = D(\alpha, \beta) D(a, \beta) \quad (27)$$

$$\begin{aligned} D(a, \alpha) D(a, \beta) &= e^{(\alpha \beta^* - \alpha^* \beta) / 2} D(a, \alpha + \beta) \\ &= D(\beta, \alpha / 2) D(a, \alpha + \beta) \\ &= D(\beta, \alpha) D(a, \beta) D(a, \alpha) \end{aligned} \quad (28)$$

$$\begin{aligned} D(a, \alpha) D(a, \beta) &= D(a, \beta) D(a, \alpha) \iff D(\beta, \alpha) = 1 \\ \iff \left(\begin{array}{c} \text{area subtended by} \\ \alpha \text{ and } \beta \end{array} \right) &= \frac{1}{2i}(\alpha \beta^* - \alpha^* \beta) = \alpha_I \beta_R - \alpha_R \beta_I = \pi k \\ \iff \left(\begin{array}{c} \text{area subtended by} \\ (\alpha_1, \alpha_2) \text{ and } (\beta_1, \beta_2) \end{array} \right) &= \alpha_2 \beta_1 - \alpha_1 \beta_2 = 2\pi k \end{aligned} \quad (29)$$

$$D(\beta, \alpha / 2) - D^*(\beta, \alpha / 2) = 2i \sin \left(i \frac{\beta \alpha^* - \beta^* \alpha}{2} \right) = -2i \sin \left(i \frac{\alpha \beta^* - \alpha^* \beta}{2} \right) \quad (30)$$

$$[D(a, \alpha), D(a, \beta)] = [D(\beta, \alpha / 2) - D^*(\beta, \alpha / 2)] D(a, \alpha + \beta) = -2i \sin \left(i \frac{\alpha \beta^* - \alpha^* \beta}{2} \right) D(a, \alpha + \beta) \quad (31)$$

$$\begin{aligned}
D(a, \alpha)D^\dagger(a, \beta) &= e^{-(\alpha\beta^* - \alpha^*\beta)/2} D(a, \alpha - \beta) \\
&= D(\beta, -\alpha/2)D(a, \alpha - \beta) \\
&= D(\beta, -\alpha)D^\dagger(a, \beta)D(a, \alpha)
\end{aligned} \tag{32}$$

$$D^\dagger(a, \beta)D(a, \alpha) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(a, \alpha - \beta) = D(\beta, \alpha/2)D(a, \alpha - \beta) \tag{33}$$

$$e^{i\theta a^\dagger a} D(a, \alpha) e^{-i\theta a^\dagger a} = D(ae^{-i\theta}, \alpha) = D(a, \alpha e^{i\theta}) \tag{34}$$

$$\begin{aligned}
e^{\lambda a^\dagger a} D(a, \alpha) e^{-\lambda a^\dagger a} &= e^{-|\alpha|^2/2} e^{\lambda a^\dagger a} e^{\alpha a^\dagger} e^{-\lambda a^\dagger a} e^{\lambda a^\dagger a} e^{-\alpha^* a} e^{-\lambda a^\dagger a} \\
&= e^{-|\alpha|^2/2} e^{\alpha a^\dagger e^\lambda} e^{-\alpha^* a e^{-\lambda}} \\
&= e^{-|\alpha|^2/2} e^{\alpha e^\lambda a^\dagger} e^{-\alpha^* e^\lambda a} e^{2\alpha^* a \sinh \lambda} \\
&= e^{-|\alpha|^2(1-e^{2\lambda})/2} D(a, \alpha e^\lambda) e^{2\alpha^* a \sinh \lambda}
\end{aligned} \tag{35}$$

4. Number states

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \quad (36)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad a|n\rangle = \sqrt{n}|n-1\rangle \quad (37)$$

$$a^\dagger a|n\rangle = n|n\rangle \quad (38)$$

$$\langle n|m\rangle = \delta_{nm} \quad (39)$$

$$\begin{aligned} \langle n|a|n\rangle &= 0 & \langle n|a^2|n\rangle &= 0 \\ \langle n|a^\dagger a|n\rangle &= n \end{aligned} \quad (40)$$

$$\begin{aligned} \langle n|x|n\rangle &= 0 & \langle n|x^2|n\rangle &= \langle n|p^2|n\rangle = n + \frac{1}{2} \\ \langle n|p|n\rangle &= 0 & \langle n|(xp + px)|n\rangle &= 0 \end{aligned} \quad (41)$$

$$\begin{aligned} e^{-\alpha^* a}|n\rangle &= \sum_{k=0}^{\infty} \frac{(-\alpha^*)^k}{k!} a^k|n\rangle = \sum_{k=0}^n \frac{(-\alpha^*)^k}{k!} \sqrt{\frac{n!}{(n-k)!}} |n-k\rangle \\ \langle n|e^{\alpha a^\dagger} &= \sum_{k=0}^n \frac{\alpha^k}{k!} \sqrt{\frac{n!}{(n-k)!}} \langle n-k| \end{aligned} \quad (42)$$

$$\begin{aligned} m \geq n : \quad \langle m|e^{\alpha a^\dagger} e^{-\alpha^* a}|n\rangle &= \sum_{l=0}^m \sum_{k=0}^n \frac{\alpha^l (-\alpha^*)^k}{l! k!} \sqrt{\frac{m! n!}{(m-l)! (n-k)!}} \langle m-l|n-k\rangle \\ &= \sum_{k=0}^n \frac{\alpha^{m-n} (-|\alpha|^2)^k}{(m-n+k)! k!} \frac{\sqrt{m! n!}}{(n-k)!} \quad (\text{Here we use } m \geq n.) \\ &= \sqrt{\frac{n!}{m!}} \alpha^{m-n} \sum_{k=0}^n \frac{(n+m-n)!}{k! (n-k)! (m-n+k)!} (-|\alpha|^2)^k \\ &= \sqrt{\frac{n!}{m!}} \alpha^{m-n} L_n^{(m-n)}(|\alpha|^2) \end{aligned} \quad (43)$$

$[L_n^{(\alpha)}(x)]$ is the generalized Laguerre polynomial of A&S 22.2.12.]

$$\langle m|D(a, \alpha)|n\rangle = \begin{cases} \sqrt{\frac{n!}{m!}} e^{-|\alpha|^2/2} \alpha^{m-n} L_n^{(m-n)}(|\alpha|^2) , & m \geq n \\ \sqrt{\frac{m!}{n!}} e^{-|\alpha|^2/2} (-\alpha^*)^{n-m} L_m^{(n-m)}(|\alpha|^2) , & m \leq n \end{cases} \quad (44)$$

$$\langle m|D(a, \alpha^*)|n\rangle = (-1)^{m-n} \langle n|D(a, \alpha)|m\rangle \quad (45)$$

$$\langle n|D(a, \alpha)|n\rangle = e^{-|\alpha|^2/2} L_n(|\alpha|^2) \quad (46)$$

$$e^{-i\theta a^\dagger a}|n\rangle = e^{-in\theta}|n\rangle \quad (47)$$

$$\begin{aligned} 0 = \langle x|a|0\rangle &= \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right) \langle x|0\rangle \quad \Rightarrow \quad \langle x|0\rangle = \frac{1}{\pi^{1/4}} e^{-x^2/2} \\ & \quad \text{(phase chosen by convention)} \end{aligned} \quad (48)$$

$$\langle p|0\rangle = \int_{-\infty}^{\infty} dx \langle p|x\rangle \langle x|0\rangle = \frac{1}{\pi^{1/4}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ipx} e^{-x^2/2} = \frac{1}{\pi^{1/4}} e^{-p^2/2}$$

$$\begin{aligned}
\langle x|n\rangle &= \frac{1}{\sqrt{n!}} \langle x|(a^\dagger)^n|0\rangle \\
&= \frac{1}{\sqrt{2^n n!}} \langle x|(x - ip)^n|0\rangle \\
&= \frac{1}{\sqrt{2^n n!}} \left(x - \frac{d}{dx}\right)^n \langle x|0\rangle \\
&= \frac{1}{\pi^{1/4} \sqrt{2^n n!}} \underbrace{\left(x - \frac{d}{dx}\right)^n e^{-x^2/2}}_{= (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}} \\
&= \frac{e^{-x^2/2}}{\pi^{1/4} \sqrt{2^n n!}} \underbrace{(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}}_{= H_n(x)}
\end{aligned} \tag{49}$$

[Use Rodrigues's formula for the Hermite polynomial $H_n(x)$: A&S 22.11.7.]

$$\langle x|n\rangle = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} e^{-x^2/2} H_n(x) = \frac{1}{\sqrt{2^n n!}} H_n(x) \langle x|0\rangle \tag{50}$$

$$\langle x|0\rangle = \frac{1}{\pi^{1/4}} e^{-x^2/2} \tag{51}$$

5. Normal ordering

Normal ordering, denoted by paired colons, applies to functions of creation and annihilation operators, i.e., expressions written in terms of a and $a^\dagger$. It means to move all annihilation operators to the right and all creation operators to the left without regard to commutators. It is meaningless to refer to the normal-ordered form of an operator A , i.e., to write $:A:$, because the result of normal ordering depends on how A is written in terms of creation and annihilation operators. For example, if $A = aa^\dagger = a^\dagger a + 1$, the result of normal ordering the first form is $a^\dagger a$, but the result of normal ordering the second form is $a^\dagger a + 1$.

$$\begin{aligned}
:(a^\dagger a)^k: &\equiv (a^\dagger)^k a^k \\
&= \sum_{n,m=0}^{\infty} |n\rangle\langle n| (a^\dagger)^k a^k |m\rangle\langle m| \\
&= \sum_{n=k}^{\infty} |n\rangle\langle n| (a^\dagger)^k a^k |n\rangle\langle n| \\
&= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} |n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1) |n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} a^\dagger a (a^\dagger a - 1) \cdots (a^\dagger a - k + 1) |n\rangle\langle n| \\
&= a^\dagger a (a^\dagger a - 1) \cdots (a^\dagger a - k + 1) \equiv (a^\dagger a)^{(k)} = (-1)^k (-a^\dagger a)_k
\end{aligned} \tag{52}$$

The final form in Eq. (52) is called a *falling factorial*. Its expectation value is a *factorial moment*. The notation comes from the *Pochhammer symbol*: $(x)_k \equiv x(x+1)\cdots(x+k-1) = (x+k)!/x!$, which is the rising factorial. The falling factorial, $(-1)^k (-x)_k = x(x-1)\cdots(x-k+1)$, which is what we have here, is also, confusingly, sometimes denoted by the Pochhammer symbol.

$$\begin{aligned}
:f(a^\dagger a): &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} : (a^\dagger a)^k : \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} |n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| \sum_{k=0}^n \frac{n!}{k! (n-k)!} f^{(k)}(0)
\end{aligned} \tag{53}$$

$$\begin{aligned}
:e^{-\lambda a^\dagger a}: &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger)^k a^k \\
&= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} |n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| \sum_{k=0}^n \frac{n!}{k! (n-k)!} (-\lambda)^k \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| (1-\lambda)^n \\
&= (1-\lambda)^{a^\dagger a} \\
&= e^{\ln(1-\lambda) a^\dagger a}
\end{aligned} \tag{54}$$

$$\begin{aligned}
:e^{-\lambda(a^\dagger - \alpha^*)(a - \alpha)}: &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger - \alpha^*)^k (a - \alpha)^k \\
&= D(a, \alpha) \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger)^k a^k \right) D^\dagger(a, \alpha) \\
&= D(a, \alpha) :e^{-\lambda a^\dagger a}: D^\dagger(a, \alpha) \\
&= (1 - \lambda)^{(a^\dagger - \alpha^*)(a - \alpha)}
\end{aligned} \tag{55}$$

6. Antinormal ordering

Antinormal ordering, denoted here, for lack of thinking of anything better, by paired bullets, applies to functions of creation and annihilation operators, i.e., expressions written in terms of a and $a^\dagger$. It means to move all annihilation operators to the left and all creation operators to the right without regard to commutators.

$$\begin{aligned}
\bullet(a^\dagger a)^k \bullet &\equiv a^k (a^\dagger)^k \\
&= \sum_{n,m=0}^{\infty} |n\rangle \langle n| a^k (a^\dagger)^k |m\rangle \langle m| \\
&= \sum_{n=0}^{\infty} |n\rangle \langle n| a^k (a^\dagger)^k |n\rangle \langle n| \\
&= \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} |n\rangle \langle n| \\
&= \sum_{n=0}^{\infty} (n+1)(n+2)\cdots(n+k) |n\rangle \langle n| \\
&= \sum_{n=0}^{\infty} aa^\dagger(aa^\dagger+1)\cdots(aa^\dagger+k-1) |n\rangle \langle n| \\
&= aa^\dagger(aa^\dagger+1)\cdots(aa^\dagger+k-1) \equiv (aa^\dagger)_k
\end{aligned} \tag{56}$$

The final form in Eq. (56) is called a *rising factorial*. The notation comes from the *Pochhammer symbol*: $(x)_k \equiv x(x+1)\cdots(x+k-1) = (x+k)!/x!$, which is the rising factorial.

$$\begin{aligned}
\bullet f(a^\dagger a) \bullet &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \bullet (a^\dagger a)^k \bullet \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} |n\rangle \langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle \langle n| \sum_{k=0}^{\infty} \frac{(n+k)!}{k! n!} f^{(k)}(0)
\end{aligned} \tag{57}$$

$$\begin{aligned}
\bullet e^{-\lambda aa^\dagger} \bullet &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} a^k (a^\dagger)^k \\
&= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{n=0}^{\infty} (n+1)(n+2)\cdots(n+k) |n\rangle \langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle \langle n| \sum_{k=0}^{\infty} \frac{(n+1)(n+2)\cdots(n+k)}{k!} (-\lambda)^k \\
&= \sum_{n=0}^{\infty} |n\rangle \langle n| \sum_{k=0}^{\infty} \frac{[-(n+1)][-(n+1)-1]\cdots[-(n+1)-k+1]}{k!} \lambda^k \\
&= \sum_{n=0}^{\infty} |n\rangle \langle n| (1+\lambda)^{-(n+1)} \\
&= (1+\lambda)^{-aa^\dagger} \\
&= e^{-\ln(1+\lambda)aa^\dagger}
\end{aligned} \tag{58}$$

$$\begin{aligned}
\bullet e^{-\lambda(a^\dagger - \alpha^*)(a - \alpha)} \bullet &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger - \alpha^*)^k (a - \alpha)^k \\
&= D(a, \alpha) \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger)^k a^k \right) D^\dagger(a, \alpha) \\
&= D(a, \alpha) \bullet e^{-\lambda a^\dagger a} \bullet D^\dagger(a, \alpha) \\
&= (1 + \lambda)^{-(a - \alpha)(a^\dagger - \alpha^*)}
\end{aligned} \tag{59}$$

7. Coherent states

$$|\alpha\rangle \equiv D(a, \alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (60)$$

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \quad (61)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad \langle\alpha|a^\dagger = \langle\alpha|\alpha^* \quad (62)$$

$$e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-i\theta a^\dagger a}D(a, \alpha)|0\rangle = D(a, \alpha e^{-i\theta})e^{-i\theta a^\dagger a}|0\rangle = |\alpha e^{-i\theta}\rangle \quad (63)$$

$$\begin{aligned} \langle\alpha|a^2|\alpha\rangle &= \alpha^2 \\ \langle\alpha|a^\dagger a|\alpha\rangle &= |\alpha|^2 \\ \langle\alpha|a|\alpha\rangle &= \alpha \\ \langle\alpha|(a^\dagger a)^2|\alpha\rangle &= |\alpha|^4 + |\alpha|^2 \\ \langle\alpha|(\Delta a^\dagger a)^2|\alpha\rangle &= |\alpha|^2 \end{aligned} \quad (64)$$

$$\begin{aligned} \langle\alpha|x|\alpha\rangle &= \alpha_1 & \langle\alpha|(\Delta x)^2|\alpha\rangle &= \langle\alpha|(\Delta p)^2|\alpha\rangle = \frac{1}{2} \\ \langle\alpha|p|\alpha\rangle &= \alpha_2 & \langle\alpha|(\Delta x \Delta p + \Delta p \Delta x)|\alpha\rangle &= 0 \end{aligned} \quad (65)$$

$$\langle x|\alpha\rangle = \langle x|D(a, \alpha)|0\rangle = e^{-i\alpha_1\alpha_2/2} e^{i\alpha_2 x} \langle x - \alpha_1|0\rangle = \frac{e^{-i\alpha_1\alpha_2/2}}{\pi^{1/4}} e^{-(x-\alpha_1)^2/2} e^{i\alpha_2 x} \quad (66)$$

$$\langle p|\alpha\rangle = \langle p|D(a, \alpha)|0\rangle = e^{i\alpha_1\alpha_2/2} e^{-i\alpha_1 p} \langle p - \alpha_2|0\rangle = \frac{e^{i\alpha_1\alpha_2/2}}{\pi^{1/4}} e^{-(p-\alpha_2)^2/2} e^{-i\alpha_1 p}$$

$$\langle x|\alpha\rangle = \langle -x|-\alpha\rangle \quad \langle p|\alpha\rangle = \langle -p|-\alpha\rangle \quad (67)$$

$$\langle 0|D(a, \alpha)|0\rangle = \langle 0|\alpha\rangle = e^{-|\alpha|^2/2} \quad (68)$$

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(a, \beta)D(a, \alpha)|0\rangle = D(\beta, \alpha/2)e^{-|\alpha-\beta|^2/2} = e^{-|\alpha|^2/2} e^{-|\beta|^2/2} e^{\alpha\beta^*} \quad (69)$$

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2} \quad (70)$$

$$\begin{aligned} D(a, \alpha)|\beta\rangle &= D(a, \alpha)D(a, \beta)|0\rangle = D(\beta, \alpha/2)|\beta + \alpha\rangle \\ \langle\beta|D(a, \alpha) &= \langle 0|D^\dagger(a, \beta)D(a, \alpha) = \langle\beta - \alpha|D(\beta, \alpha/2) \end{aligned} \quad (71)$$

$$\langle\beta|D(a, \alpha)|\beta\rangle = e^{-|\alpha|^2/2} \langle\beta|e^{\alpha a^\dagger} e^{-\alpha^* a}|\beta\rangle = e^{-|\alpha|^2/2} D(\beta, \alpha) \quad (72)$$

$$\begin{aligned} \langle\gamma|D(a, \alpha)|\beta\rangle &= e^{-|\alpha|^2/2} \langle\gamma|e^{\alpha a^\dagger} e^{-\alpha^* a}|\beta\rangle \\ &= e^{-|\alpha|^2/2} e^{\alpha\gamma^* - \alpha^*\beta} \langle\gamma|\beta\rangle \\ &= e^{-|\alpha|^2/2} e^{-|\beta|^2/2} e^{-|\gamma|^2/2} e^{\alpha\gamma^* - \alpha^*\beta + \beta\gamma^*} \end{aligned} \quad (73)$$

$$\langle\gamma|D(a, \alpha)|\beta\rangle = D(\gamma, \alpha/2)D(\beta, \alpha/2)D(\gamma, \beta/2)e^{-|\alpha+\beta-\gamma|^2/2} \quad (74)$$

$$e^{-\lambda a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-\lambda})^n}{\sqrt{n!}} |n\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2} |\alpha e^{-\lambda}\rangle \quad (75)$$

$$e^{-\lambda a^\dagger a}|\alpha\rangle = e^{-\lambda a^\dagger a}D(a, \alpha)|0\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2} D(a, \alpha e^{-\lambda})|0\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2} |\alpha e^{-\lambda}\rangle \quad (76)$$

$$\langle\alpha|e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2(1-e^{-i\theta})} \quad (77)$$

$$\langle\alpha|e^{-\lambda a^\dagger a}|\alpha\rangle = \langle\alpha|:e^{-(1-e^{-\lambda})a^\dagger a}:|\alpha\rangle = e^{-(1-e^{-\lambda})|\alpha|^2} \quad (78)$$

$$\begin{aligned}
\alpha &= \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) = |\alpha|e^{i\phi} \\
d^2\alpha &= d\alpha_R d\alpha_I = \frac{d\alpha_1 d\alpha_2}{2} = |\alpha|d|\alpha| d\phi = \frac{1}{2}d|\alpha|^2 d\phi \\
\delta^2(\alpha) &= \delta(\alpha_R)\delta(\alpha_I) = 2\delta(\alpha_1)\delta(\alpha_2)
\end{aligned} \tag{79}$$

$$\begin{aligned}
\langle n | \left(\int d^2\alpha |\alpha\rangle\langle\alpha| \right) | m \rangle &= \int d^2\alpha \langle n | \alpha \rangle \langle \alpha | m \rangle \\
&= \frac{1}{\sqrt{n!m!}} \int d^2\alpha e^{-|\alpha|^2} \alpha^n (\alpha^*)^m \\
&= \frac{1}{2\sqrt{n!m!}} \int d|\alpha|^2 d\phi e^{-|\alpha|^2} |\alpha|^{n+m} e^{i(n-m)\phi} \\
&= \frac{\pi}{n!} \delta_{nm} \int_0^\infty du e^{-u} u^n \\
&= \pi \delta_{nm}
\end{aligned} \tag{80}$$

$$1 = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi} D(a, \alpha) |0\rangle\langle 0| D^\dagger(a, \alpha) \tag{81}$$

$$\text{tr}(A) = \int \frac{d^2\alpha}{\pi} \langle \alpha | A | \alpha \rangle = \int \frac{d^2\alpha}{\pi} \langle 0 | D^\dagger(a, \alpha) A D(a, \alpha) | 0 \rangle \tag{82}$$

$$\mathcal{I} = 1 \odot 1 = \sum_{n,m} |n\rangle\langle n| \odot |m\rangle\langle m| = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} |\alpha\rangle\langle\alpha| \odot |\beta\rangle\langle\beta| \tag{83}$$

$$\mathbf{I} = \mathcal{I}^\# = \sum_{n,m} |n\rangle\langle m| \odot |m\rangle\langle n| = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} |\alpha\rangle\langle\beta| \odot |\beta\rangle\langle\alpha|$$

$$\begin{aligned}
\mathcal{I}(A) &= A, & \mathcal{I}|A) &= 1\text{tr}(A) \\
\mathbf{I}(A) &= 1\text{tr}(A), & \mathbf{I}|A) &= A
\end{aligned} \tag{84}$$

8. Parity

$$\begin{aligned} P^\dagger x P &= -x \\ P^\dagger p P &= -p \end{aligned} \iff P^\dagger a P = -a \quad (85)$$

$$a P|0\rangle = -P a|0\rangle = 0 \implies P|0\rangle = e^{i\delta}|0\rangle = |0\rangle \quad (\text{choose phase } \delta = 0) \quad (86)$$

$$P|n\rangle = \frac{1}{\sqrt{n!}} P(a^\dagger)^n |0\rangle = \frac{(-1)^n}{\sqrt{n!}} (a^\dagger)^n P|0\rangle = (-1)^n |n\rangle \quad (87)$$

$$P = \sum_{n=0}^{\infty} (-1)^n |n\rangle \langle n| = (-1)^{a^\dagger a} = P^\dagger \quad (88)$$

$$PD(a, \alpha)P = D(a, -\alpha) = D^\dagger(a, \alpha) \quad (89)$$

$$P|\alpha\rangle = PD(a, \alpha)|0\rangle = D(a, -\alpha)P|0\rangle = D(a, -\alpha)|0\rangle = |-\alpha\rangle \quad (90)$$

$$P = \int \frac{d^2\alpha}{\pi} P|\alpha\rangle \langle \alpha| = \int \frac{d^2\alpha}{\pi} |-\alpha\rangle \langle \alpha| \quad (91)$$

$$P|x\rangle = \int \frac{d^2\alpha}{\pi} P|\alpha\rangle \langle \alpha|x\rangle = \int \frac{d^2\alpha}{\pi} |-\alpha\rangle \langle -\alpha|-x\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha|-x\rangle = |-x\rangle \quad (92)$$

$$P|p\rangle = \int \frac{d^2\alpha}{\pi} P|\alpha\rangle \langle \alpha|p\rangle = \int \frac{d^2\alpha}{\pi} |-\alpha\rangle \langle -\alpha|-p\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha|-p\rangle = |-p\rangle$$

$$P = \int dx P|x\rangle \langle x| = \int dx |-x\rangle \langle x| \quad (93)$$

$$P = \int dp P|p\rangle \langle p| = \int dp |-p\rangle \langle p|$$

$$\begin{aligned} \text{tr}(PD(a, \alpha)) &= \int \frac{d^2\beta}{\pi} \langle \beta|PD(a, \alpha)|\beta\rangle \\ &= \int \frac{d^2\beta}{\pi} \langle -\beta|D(a, \alpha)|\beta\rangle \\ &= e^{-|\alpha|^2/2} \int \frac{d^2\beta}{\pi} e^{-\alpha\beta^* - \alpha^*\beta} \langle -\beta|\beta\rangle \\ &= e^{-|\alpha|^2/2} \underbrace{\int \frac{d^2\beta}{\pi} e^{-2|\beta|^2} e^{-\beta\alpha^* - \beta^*\alpha}}_{= \frac{1}{2}e^{|\alpha|^2/2}} \\ &= \frac{1}{2} \end{aligned} \quad (94)$$

9. Fourier transform pairs

$$\int \frac{d^2\beta}{\pi} D(\beta, \alpha) = \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta} = \int \frac{d\beta_1 d\beta_2}{2\pi} e^{i(\alpha_2\beta_1 - \alpha_1\beta_2)} = 2\pi\delta(\alpha_2)\delta(\alpha_1) = \pi\delta^2(\alpha) \quad (95)$$

$$f(\alpha) = \int \frac{d^2\beta}{\pi} \tilde{f}(\beta) D(\beta, \alpha) \quad \tilde{f}(\beta) = \int \frac{d^2\alpha}{\pi} f(\alpha) D(\alpha, \beta) \quad (96)$$

$$g(\alpha) = f^*(\alpha) \quad \Longleftrightarrow \quad \tilde{g}(\beta) = \tilde{f}^*(-\beta) \quad (97)$$

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial \beta} &= \int \frac{d^2\alpha}{\pi} \alpha^* f(\alpha) D(\alpha, \beta) & \frac{\partial^2 \tilde{f}}{\partial \beta \partial \beta^*} &= - \int \frac{d^2\alpha}{\pi} |\alpha|^2 f(\alpha) D(\alpha, \beta) \\ \frac{\partial \tilde{f}}{\partial \beta^*} &= - \int \frac{d^2\alpha}{\pi} \alpha f(\alpha) D(\alpha, \beta) \end{aligned} \quad (98)$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} f(\alpha) g(\alpha) D(\alpha, \beta) &= \int \frac{d^2\gamma}{\pi} \tilde{f}(\gamma) \tilde{g}(\beta - \gamma) \\ \int \frac{d^2\beta}{\pi} \left(\int \frac{d^2\gamma}{\pi} \tilde{f}(\gamma) \tilde{g}(\beta - \gamma) \right) D(\beta, \alpha) &= f(\alpha) g(\alpha) \end{aligned} \quad (99)$$

$$\int \frac{d^2\alpha}{\pi} f(\alpha) g(\alpha) = \int \frac{d^2\gamma}{\pi} \tilde{f}(\gamma) \tilde{g}(-\gamma) \quad (100)$$

$$\int \frac{d^2\alpha}{\pi} |f(\alpha)|^2 D(\alpha, \beta) = \int \frac{d^2\gamma}{\pi} \tilde{f}(\gamma) \tilde{f}^*(\gamma - \beta) \quad (\text{Parseval's relation}) \quad (101)$$

$$\int \frac{d^2\alpha}{\pi} |f(\alpha)|^2 = \int \frac{d^2\beta}{\pi} |\tilde{f}(\beta)|^2 \quad (102)$$

10. Gaussian integrals

$$\int_{-\infty}^{\infty} du e^{-ax^2} e^{bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \quad (103)$$

$$\int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} D(\alpha, \beta) = \int \frac{d\alpha_1 d\alpha_2}{2\pi} e^{-(\alpha_1^2 + \alpha_2^2)/2} e^{i(\beta_2\alpha_1 - \beta_1\alpha_2)} = e^{-(\beta_1^2 + \beta_2^2)/2} = e^{-|\beta|^2} \quad (104)$$

$$\int \frac{d^2\alpha}{\pi\sigma^2} e^{-|\alpha|^2/\sigma^2} D(\alpha, \beta) = \int \frac{d^2\alpha'}{\pi} e^{-|\alpha'|^2} D(\alpha', \sigma\beta) = e^{-\sigma^2|\beta|^2} \quad (105)$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} e^{\alpha\gamma^* - \alpha^*\beta} &= \int \frac{d\alpha_1 d\alpha_2}{2\pi} e^{-(\alpha_1^2 + \alpha_2^2)/2} e^{\alpha_1(\gamma^* - \beta)/\sqrt{2}} e^{i\alpha_2(\gamma^* + \beta)/\sqrt{2}} \\ &= e^{(\gamma^* - \beta)^2/4} e^{-(\gamma^* + \beta)^2/4} \\ &= e^{-\beta\gamma^*} \end{aligned} \quad (106)$$

$$\int \frac{d^2\alpha}{\pi\sigma^2} e^{-|\alpha|^2/\sigma^2} e^{\alpha\gamma^* - \alpha^*\beta} = \int \frac{d^2\alpha'}{\pi} e^{-|\alpha'|^2} e^{\alpha'\sigma\gamma^* - \alpha'^*\sigma\beta} = e^{-\sigma^2\beta\gamma^*} \quad (107)$$

11. Orthogonality and completeness of displacement operators

$$\text{tr}(D(a, \alpha)) = \int \frac{d^2\beta}{\pi} \langle \beta | D(a, \alpha) | \beta \rangle = e^{-|\alpha|^2/2} \int \frac{d^2\beta}{\pi} D(\beta, \alpha) = \pi \delta^2(\alpha) \quad (108)$$

$$\text{tr}(D^\dagger(a, \beta) D(a, \alpha)) = D(\beta, \alpha/2) \text{tr}(D(a, \alpha - \beta)) = \pi \delta^2(\alpha - \beta) \quad (109)$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} \langle \mu | D(a, \alpha) | \beta \rangle \langle \gamma | D^\dagger(a, \alpha) | \nu \rangle &= \langle \mu | \beta \rangle \langle \gamma | \nu \rangle \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} e^{\alpha(\mu^* - \gamma^*) - \alpha^*(\beta - \nu)} \\ &= \langle \mu | \beta \rangle \langle \gamma | \nu \rangle e^{-(\beta - \nu)(\mu^* - \gamma^*)} \\ &= e^{-|\mu|^2/2} e^{-|\beta|^2/2} e^{\beta\mu^*} \\ &\quad \times e^{-|\gamma|^2/2} e^{-|\nu|^2/2} e^{\nu\gamma^*} e^{-(\beta - \nu)(\mu^* - \gamma^*)} \\ &= e^{-|\beta|^2/2} e^{-|\gamma|^2/2} e^{\beta\gamma^*} e^{-|\nu|^2/2} e^{-|\mu|^2/2} e^{\nu\mu^*} \\ &= \langle \gamma | \beta \rangle \langle \mu | \nu \rangle \end{aligned} \quad (110)$$

$$\begin{aligned} \text{tr}\left((a^\dagger)^k a^l \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| D(\alpha, \beta)\right) &= \int \frac{d^2\alpha}{\pi} (\alpha^*)^k \alpha^l D(\alpha, \beta) \\ &= \frac{\partial^{k+l}}{\partial \beta^k \partial (-\beta^*)^l} \underbrace{\int \frac{d^2\alpha}{\pi} D(\alpha, \beta)}_{= \pi \delta^2(\beta) = \text{tr}(D(a, \beta))} \\ &= \frac{\partial^{k+l}}{\partial \beta^k \partial (-\beta^*)^l} \text{tr}(e^{-\beta^* a} e^{\beta a^\dagger}) \\ &= \text{tr}((a^\dagger)^k a^l e^{-\beta^* a} e^{\beta a^\dagger}) \end{aligned} \quad (111)$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| D(\alpha, \beta) &= e^{-\beta^* a} e^{\beta a^\dagger} = e^{-|\beta|^2/2} D(a, \beta) \\ \int \frac{d^2\beta}{\pi} e^{-|\beta|^2/2} D(a, \beta) D(\beta, \alpha) &= |\alpha\rangle \langle \alpha| \end{aligned} \quad (112)$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} D(a, \alpha) |\beta\rangle \langle \gamma| D^\dagger(a, \alpha) &= \int \frac{d^2\alpha}{\pi} D(a, \alpha) D(a, \beta) |0\rangle \langle 0| D^\dagger(a, \gamma) D^\dagger(a, \alpha) \\ &= D(a, \beta) \left(\int \frac{d^2\alpha}{\pi} D(a + \beta, \alpha) |0\rangle \langle 0| D^\dagger(a + \gamma, \alpha) \right) D^\dagger(a, \gamma) \\ &= D(a, \beta) \left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| D(\alpha, \gamma - \beta) \right) D^\dagger(a, \gamma) \\ &= e^{-|\gamma - \beta|^2/2} D(a, \beta) D(a, \gamma - \beta) D^\dagger(a, \gamma) \\ &= D(\gamma, \beta/2) e^{-|\beta - \gamma|^2/2} D(a, \beta) D^\dagger(a, \beta) D(a, \gamma) D^\dagger(a, \gamma) \\ &= \langle \gamma | \beta \rangle 1 \end{aligned} \quad (113)$$

$$\mathbf{I} = \int \frac{d^2\alpha}{\pi} D(a, \alpha) \odot D^\dagger(a, \alpha) = \int \frac{d^2\alpha}{\pi} D^\dagger(a, \alpha) \odot D(a, \alpha) \quad (114)$$

[Follows from Eq. (110) or Eq. (113)]

$$\text{tr}(A) 1 = \mathbf{I}(A) = \int \frac{d^2\alpha}{\pi} D(a, \alpha) A D^\dagger(a, \alpha) \quad (115)$$

$$1 = \int \frac{d^2\alpha}{\pi} D(a, \alpha) \rho D^\dagger(a, \alpha) = \int \frac{d^2\alpha}{\pi} D(a, \alpha) |\psi\rangle \langle \psi| D^\dagger(a, \alpha) \quad (116)$$

$$\mathrm{tr}(A) = \int \frac{d^2\alpha}{\pi} \mathrm{tr}(D(a, \alpha) \rho D^\dagger(a, \alpha) A) = \int \frac{d^2\alpha}{\pi} \langle \psi | D^\dagger(a, \alpha) A D(a, \alpha) | \psi \rangle \quad (117)$$

12. Operator ordering

$$D^{(s)}(a, \alpha) \equiv e^{s|\alpha|^2/2} D(a, \alpha) \quad (118)$$

$$D^{(s)\dagger}(a, \alpha) = D^{(s)}(-a, \alpha) = D^{(s)}(a, -\alpha) \quad (119)$$

$$\text{tr}(D^{(s)}(a, \alpha)) = \pi \delta^2(\alpha) \quad (120)$$

$$\text{tr}(D^{(-s)\dagger}(a, \beta) D^{(s)}(a, \alpha)) = \pi \delta^2(\alpha - \beta) \quad (121)$$

$$\mathbf{I} = \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) \odot D^{(s)}(a, \alpha) \quad (122)$$

$$s = +1 \text{ (normal ordering): } D^{(+1)}(a, \alpha) = e^{\alpha a^\dagger} e^{-\alpha^* a} = :D(a, \alpha):$$

$$s = 0 \text{ (symmetric ordering): } D^{(0)}(a, \alpha) = D(a, \alpha) = e^{\alpha a^\dagger - \alpha^* a} \quad (123)$$

$$s = -1 \text{ (antinormal ordering): } D^{(-1)}(a, \alpha) = e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$[(a^\dagger)^k a^l]_{(s)} \equiv \left. \frac{\partial^{k+l} D^{(s)}(a, \alpha)}{\partial \alpha^k \partial (-\alpha^*)^l} \right|_{\alpha=0} \quad (124)$$

$$D^{(s)}(a, \alpha) = \sum_{k,l} \frac{\alpha^k (-\alpha^*)^l}{k! l!} [(a^\dagger)^k a^l]_{(s)}$$

$$[(a^\dagger)^k a^l]_{(+1)} = (a^\dagger)^k a^l \quad (125)$$

$$[(a^\dagger)^k a^l]_{(-1)} = a^l (a^\dagger)^k$$

$$\begin{aligned} [(a^\dagger)^k a^k]_{(s)} &= \left. \frac{\partial^{2k} [e^{(s-1)\alpha\alpha^*/2} D^{(+1)}(a, \alpha)]}{\partial \alpha^k \partial (-\alpha^*)^k} \right|_{\alpha=\alpha^*=0} \\ &= \sum_{m=0}^k \left(\frac{k!}{m!(k-m)!} \right)^2 \underbrace{\left. \frac{\partial^{2m} e^{-(1-s)\alpha\alpha^*/2}}{\partial \alpha^m \partial (-\alpha^*)^m} \right|_{\alpha=\alpha^*=0}}_{= m! \left. \frac{d^m e^{(1-s)x/2}}{dx^m} \right|_{x=0}} \underbrace{\left. \frac{\partial^{2(k-m)} D^{(+1)}(a, \alpha)}{\partial \alpha^{k-m} \partial (-\alpha^*)^{k-m}} \right|_{\alpha=\alpha^*=0}}_{= (a^\dagger)^{k-m} a^{k-m}} \\ &= \sum_{m=0}^k \frac{1}{m!} \left(\frac{k!}{(k-m)!} \right)^2 \left(\frac{1-s}{2} \right)^m (a^\dagger)^{k-m} a^{k-m} \\ &= \sum_{m=0}^k \frac{1}{m!} \left(\frac{k!}{(k-m)!} \right)^2 \left(\frac{1-s}{2} \right)^m (a^\dagger a)^{(k-m)} \\ &= \left(\frac{1-s}{2} \right)^k k! \sum_{m=0}^k \frac{k!}{m!(k-m)!} \frac{(a^\dagger a)^{(m)}}{m!} \left(\frac{2}{1-s} \right)^m \end{aligned} \quad (126)$$

$$\begin{aligned} \tilde{D}^{(s)}(a, \beta) &\equiv \int \frac{d^2\alpha}{\pi} D^{(s)}(a, \alpha) D(\alpha, \beta) = \int \frac{d^2\alpha}{\pi} D^{(s)}(a - \beta, \alpha) \equiv \delta^{(s)2}(a - \beta) \\ D^{(s)}(a, \alpha) &= \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) D(\beta, \alpha) \end{aligned} \quad (127)$$

In general lingo, the operators $D^{(s)}(a, \alpha)$ are *frame operators*, and the Fourier transform is their expression in terms of *harmonic tensors*, which in this case are displacement operators.

$$\tilde{D}^{(s)\dagger}(a, \beta) = \tilde{D}^{(s)}(a, \beta) \quad (128)$$

$$\text{tr}(\tilde{D}^{(s)}(a, \alpha)) = 1 \quad (129)$$

$$\int d^2\beta \tilde{D}^{(s)}(a, \beta) = D^{(s)}(a, 0) = 1 \quad (130)$$

$$\begin{aligned} \text{tr}(\tilde{D}^{(-s)\dagger}(a, \gamma)\tilde{D}^{(s)}(a, \beta)) &= \int \frac{d^2\alpha'}{\pi} \frac{d^2\alpha}{\pi} \underbrace{\text{tr}(D^{(-s)\dagger}(a, \alpha')D^{(s)}(a, \alpha))}_{=\pi\delta^2(\alpha - \alpha')} D(\alpha', -\gamma)D(\alpha, \beta) \\ &= \int \frac{d^2\alpha}{\pi} D(\alpha, \beta - \gamma) \\ &= \pi\delta^2(\beta - \gamma) \end{aligned} \quad (131)$$

$$\begin{aligned} \mathbf{I} &= \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) \odot D^{(s)}(a, \alpha) = \int \frac{d^2\alpha}{\pi} D^{(-s)}(a, \alpha) \odot D^{(s)\dagger}(a, \alpha) \\ &= \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) \odot \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta)D(\beta, \alpha) \\ &= \int \frac{d^2\beta}{\pi} \left(\int \frac{d^2\alpha}{\pi} D^{(-s)}(a, \alpha)D(\alpha, \beta) \right)^\dagger \odot \tilde{D}^{(s)}(a, \beta) \\ &= \int \frac{d^2\beta}{\pi} \tilde{D}^{(-s)}(a, \beta) \odot \tilde{D}^{(s)}(a, \beta) \end{aligned} \quad (132)$$

$$\begin{aligned} [(a^\dagger)^k a^l]_{(s)} &= \frac{\partial^{k+l} D^{(s)}(a, \alpha)}{\partial \alpha^k \partial (-\alpha^*)^l} \Big|_{\alpha=0} \\ &= \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) \underbrace{\frac{\partial^{k+l} D(\beta, \alpha)}{\partial \alpha^k \partial (-\alpha^*)^l} \Big|_{\alpha=0}}_{=(\beta^*)^k \beta^l} \\ &= \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) (\beta^*)^k \beta^l \\ &= \left([(a^\dagger)^l a^k]_{(s)} \right)^\dagger \end{aligned} \quad (133)$$

$$\begin{aligned} \tilde{D}^{(s)}(a, \beta) &= \int \frac{d^2\alpha}{\pi} D^{(s)}(a, \alpha)D(\alpha, \beta) \\ &= \int \frac{d^2\alpha}{\pi} e^{(s-s')|\alpha|^2/2} D^{(s')}(a, \alpha)D(\alpha, \beta) \\ &= \int \frac{d^2\alpha}{\pi} e^{-(s'-s)|\alpha|^2/2} D(\alpha, \beta) \int \frac{d^2\gamma}{\pi} \tilde{D}^{(s')}(a, \gamma)D(\gamma, \alpha) \\ &= \int \frac{d^2\gamma}{\pi} \tilde{D}^{(s')}(a, \gamma) \underbrace{\int \frac{d^2\alpha}{\pi} e^{-(s'-s)|\alpha|^2/2} D(\alpha, \beta - \gamma)}_{=\frac{2}{s'-s} e^{-2|\beta-\gamma|^2/(s'-s)}} \\ &= \frac{2}{s'-s} \int \frac{d^2\gamma}{\pi} \tilde{D}^{(s')}(a, \gamma) e^{-2|\gamma-\beta|^2/(s'-s)}, \quad s \leq s' \end{aligned} \quad (134)$$

$$\begin{aligned} \text{tr}(\tilde{D}^{(-s)}(a, \beta)[(a^\dagger)^k a^l]_{(s)}) &= \int \frac{d^2\gamma}{\pi} \underbrace{\text{tr}(\tilde{D}^{(-s)\dagger}(a, \beta)\tilde{D}^{(s)}(a, \gamma))}_{=\pi\delta^2(\beta - \gamma)} (\gamma^*)^k \gamma^l = (\beta^*)^k \beta^l \end{aligned} \quad (135)$$

$$\text{tr}(\tilde{D}^{(-s)}(a, \beta) D^{(s)}(a, \alpha)) = \int \frac{d^2\gamma}{\pi} \underbrace{\text{tr}(\tilde{D}^{(-s)\dagger}(a, \beta) \tilde{D}^{(s)}(a, \gamma))}_{=\pi\delta^2(\beta - \gamma)} D(\gamma, \alpha) = D(\beta, \alpha) \quad (136)$$

$$\begin{aligned} \tilde{D}^{(s)}(a, \beta) &= \int \frac{d^2\alpha}{\pi} D^{(s)}(a - \beta, \alpha) \\ &= D(a, \beta) \left(\int \frac{d^2\alpha}{\pi} D^{(s)}(a, \alpha) \right) D^\dagger(a, \beta) \\ &= D(a, \beta) \tilde{D}^{(s)}(a, 0) D^\dagger(a, \beta) \end{aligned} \quad (137)$$

$$\begin{aligned} \tilde{D}^{(s)}(a, \beta + \gamma) &= D(a, \beta + \gamma) \tilde{D}^{(s)}(a, 0) D^\dagger(a, \beta + \gamma) \\ &= D(a, \beta) D(a, \gamma) \tilde{D}^{(s)}(a, 0) D^\dagger(a, \gamma) D^\dagger(a, \beta) \\ &= D(a, \beta) \tilde{D}^{(s)}(a, \gamma) D^\dagger(a, \beta) \end{aligned} \quad (138)$$

$$\begin{aligned} \tilde{D}^{(s)}(a, \beta) &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} \tilde{D}^{(s')}(a, \gamma) e^{-2|\gamma - \beta|^2/(s' - s)} \\ &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} \tilde{D}^{(s')}(a, \gamma + \beta) e^{-2|\gamma|^2/(s' - s)} \\ &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} D(a, \gamma) \tilde{D}^{(s')}(a, \beta) D^\dagger(a, \gamma) e^{-2|\gamma|^2/(s' - s)}, \quad s \leq s' \end{aligned} \quad (139)$$

$$\begin{aligned} \langle \alpha | \tilde{D}^{(s)}(a, 0) | \beta \rangle &= \int \frac{d^2\gamma}{\pi} \langle \alpha | D^{(s)}(a, \gamma) | \beta \rangle \\ &= \langle \alpha | \beta \rangle \int \frac{d^2\gamma}{\pi} e^{-(1-s)|\gamma|^2/2} e^{\gamma\alpha^* - \gamma^*\beta} \\ &= \langle \alpha | \beta \rangle \frac{2}{1-s} e^{-2\beta\alpha^*/(1-s)} \quad s < 1, \\ &= \frac{2}{1-s} \langle \alpha | e^{-\mu a^\dagger a} | \beta \rangle, \quad e^\mu = \frac{s-1}{s+1} \end{aligned} \quad (140)$$

$$\tilde{D}^{(s)}(a, 0) = \frac{1}{Z} e^{-\mu a^\dagger a} = \frac{2}{1-s} \left(\frac{s+1}{s-1} \right)^{a^\dagger a}, \quad Z = \frac{1-s}{2} = \frac{1}{1-e^{-\mu}}, \quad s = -\coth(\mu/2) \quad (141)$$

As s goes from $-\infty$ to -1_- , μ ranges over positive inverse temperatures, going from $\mu = 0_+$ at $s = -\infty$ to $\mu = \infty$ at $s = -1_-$. At $s = -1$, $\mu = \nu \pm i\pi$ acquires an imaginary part $\pm i\pi$; as s ranges from -1_+ to 1_- , ν ranges over the whole real line, going from $\nu = \infty$ at $s = -1_+$ to $\nu = 0$ at $s = 0$ and to $\nu = -\infty$ at $s = 1_-$. At $s = +1$, μ becomes real again; as s ranges from 1_+ to ∞ , μ ranges over negative inverse temperatures, going from $\mu = -\infty$ at $s = 1_+$ to $\mu = 0_-$ at $s = \infty$.

$$\begin{aligned}
\tilde{D}^{(0)}(a, \gamma) \tilde{D}^{(0)}(a, \beta) &= \tilde{D}^{(0)\dagger}(a, \gamma) \tilde{D}^{(0)}(a, \beta) \\
&= \int \frac{d^2\alpha'}{\pi} \frac{d^2\alpha}{\pi} D(\alpha', -\gamma) D(\alpha, \beta) \underbrace{D^\dagger(a, \alpha') D(a, \alpha)}_{= D(\alpha', \alpha/2) D(a, \alpha - \alpha')} \\
&= \int \frac{d^2\mu}{\pi} D(a, \mu) \int \frac{d^2\nu}{\pi} D(\nu, \mu/2) D(\nu - \mu/2, -\gamma) D(\nu + \mu/2, \beta) \\
&= \int \frac{d^2\mu}{\pi} D(a, \mu) D(\mu, (\beta + \gamma)/2) \underbrace{\int \frac{d^2\nu}{\pi} D(\nu, \mu/2 + \beta - \gamma)}_{= 4\pi\delta^2(\mu - 2(\gamma - \beta))} \\
&= 4D(a, 2(\gamma - \beta)) D(\gamma - \beta, \beta + \gamma) \\
&= 4D(a, 2(\gamma - \beta)) D(\gamma, 2\beta) \\
&= 4D^\dagger(a, -2\gamma) D(a, -2\beta)
\end{aligned} \tag{142}$$

$$\begin{aligned}
[\tilde{D}^{(0)}(a, \alpha), D(a, \beta)] &= \int \frac{d^2\gamma}{\pi} D(\gamma, \alpha) [D(a, \gamma), D(a, \beta)] \\
&= \int \frac{d^2\gamma}{\pi} (D(\gamma, \alpha - \beta/2) - D(\gamma, \alpha + \beta/2)) D(a, \gamma + \beta) \\
&= D(\alpha, \beta) \int \frac{d^2\gamma}{\pi} D(a, \gamma) (D(\gamma, \alpha - \beta/2) - D(\gamma, \alpha + \beta/2)) \\
&= D(\alpha, \beta) (\tilde{D}^{(0)}(a, \alpha - \beta/2) - \tilde{D}^{(0)}(a, \alpha + \beta/2))
\end{aligned} \tag{143}$$

$$\begin{aligned}
[\tilde{D}^{(0)}(a, \alpha), \tilde{D}^{(0)}(a, \beta)] &= \int \frac{d^2\gamma}{\pi} D(\gamma, \beta) [\tilde{D}^{(0)}(a, \alpha), D(a, \gamma)] \\
&= \int \frac{d^2\gamma}{\pi} D(\gamma, \beta) D(\alpha, \gamma) (\tilde{D}^{(0)}(a, \alpha - \gamma/2) - \tilde{D}^{(0)}(a, \alpha + \gamma/2)) \\
&= \int \frac{d^2\gamma}{\pi} \tilde{D}^{(0)}(a, \alpha + \gamma/2) (D(\gamma, \alpha - \beta) - D(\gamma, \beta - \alpha)) \\
&= 4 \int \frac{d^2\gamma}{\pi} \tilde{D}^{(0)}(a, \gamma)
\end{aligned} \tag{144}$$

$$\begin{aligned}
&\times (D(\gamma, 2(\alpha - \beta)) D(\alpha, 2\beta) - D(\gamma, 2(\beta - \alpha)) D(\alpha, -2\beta)) \\
&= 4 [D(a, 2(\alpha - \beta)) D(\alpha, 2\beta) - D(a, 2(\beta - \alpha)) D(\alpha, -2\beta)] \\
[\tilde{D}^{(0)}(a, \alpha), \tilde{D}^{(0)}(a, \beta)] &= 4D(a, 2(\alpha - \beta)) D(\alpha, 2\beta) - 4D(a, 2(\beta - \alpha)) D(\beta, 2\alpha) \\
&= 4 [D(a, 2(\alpha - \beta)) D(\alpha, 2\beta) - D(a, 2(\beta - \alpha)) D(\alpha, -2\beta)]
\end{aligned} \tag{145}$$

$$s = 0: \begin{cases} \langle \alpha | \tilde{D}^{(0)}(a, 0) | \beta \rangle = \langle \alpha | \beta \rangle 2e^{-2\beta\alpha^*} = 2\langle \alpha | -\beta \rangle \implies \tilde{D}^{(0)}(a, 0) = 2P \\ \tilde{D}^{(0)}(a, \beta) = 2D(a, \beta) P D^\dagger(a, \beta) = 2P D^\dagger(a, 2\beta) = 2P D(a, -2\beta) \\ \langle \alpha | \tilde{D}^{(0)}(a, \gamma) | \beta \rangle = 2\langle -\alpha | D(a, -2\gamma) | \beta \rangle = 2D(\gamma, \beta - \alpha) D(\beta, \alpha/2) e^{-2|\gamma - (\alpha + \beta)/2|^2} \\ \tilde{D}^{(0)}(a, \beta) = 2 \int \frac{d^2\alpha}{\pi} D(a, \beta) |\alpha\rangle \langle -\alpha | D^\dagger(a, \beta) = 2 \int \frac{d^2\alpha}{\pi} |\beta + \alpha\rangle \langle \beta - \alpha | D(\alpha, \beta) \end{cases} \tag{146}$$

$$\begin{aligned}
\langle x | \tilde{D}^{(0)}(a, \beta) | x' \rangle &= \langle x | 2P D(a, -2\beta) | x' \rangle = 2\langle -x | D(a, -2\beta) | x' \rangle \\
&= e^{-2i\beta_1\beta_2} e^{2i\beta_2x} \delta(\beta_1 - (x + x')/2) \\
&= e^{i\beta_2(x - x')} \delta(\beta_1 - (x + x')/2)
\end{aligned} \tag{147}$$

$$\begin{aligned}
\langle p|\tilde{D}^{(0)}(a, \beta)|p'\rangle &= \langle p|2PD(a, -2\beta)|p'\rangle = 2\langle -p|D(a, -2\beta)|p'\rangle \\
&= e^{2i\beta_1\beta_2}e^{-2i\beta_1p}\delta(\beta_2 - (p+p')/2) \\
&= e^{-i\beta_1(p-p')}\delta(\beta_2 - (p+p')/2)
\end{aligned} \tag{148}$$

$$s = -1: \begin{cases} \langle \alpha|\tilde{D}^{(-1)}(a, 0)|\beta\rangle = \langle \alpha|\beta\rangle e^{-\beta\alpha^*} = e^{-|\alpha|^2/2}e^{-|\beta|^2/2} = \langle \alpha|0\rangle\langle 0|\beta\rangle \\ \implies \tilde{D}^{(-1)}(a, 0) = |0\rangle\langle 0| \\ \tilde{D}^{(-1)}(a, \beta) = D(a, \beta)|0\rangle\langle 0|D^\dagger(a, \beta) = |\beta\rangle\langle \beta| \\ |\beta\rangle\langle \beta| = \int \frac{d^2\alpha}{\pi} e^{-\alpha^*a}e^{\alpha a^\dagger} D(\alpha, \beta) \quad e^{-\alpha^*a}e^{\alpha a^\dagger} = \int \frac{d^2\beta}{\pi} |\beta\rangle\langle \beta|D(\beta, \alpha) \end{cases} \tag{149}$$

$$\begin{aligned}
\mathcal{G}^{(s)} &= \int \frac{d^2\alpha}{\pi} D^{(s)\dagger}(a, \alpha) \odot D^{(s)}(a, \alpha) = \int \frac{d^2\alpha}{\pi} D^{(s)}(a, \alpha) \odot D^{(s)\dagger}(a, \alpha) \\
&= \int \frac{d^2\alpha}{\pi} e^{s|\alpha|^2} D^\dagger(a, \alpha) \odot D(a, \alpha) = \int \frac{d^2\alpha}{\pi} e^{s|\alpha|^2} D(a, \alpha) \odot D^\dagger(a, \alpha) \\
&= \int \frac{d^2\alpha}{\pi} D^{(s)\dagger}(a, \alpha) \odot \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta)D(\beta, \alpha) \\
&= \int \frac{d^2\beta}{\pi} \left(\int \frac{d^2\alpha}{\pi} D^{(s)}(a, \alpha)D(\alpha, \beta) \right)^\dagger \odot \tilde{D}^{(s)}(a, \beta) \\
&= \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) \odot \tilde{D}^{(s)}(a, \beta)
\end{aligned} \tag{150}$$

$\mathcal{G}^{(s)}$ is the *quartic twirl* of the frame operators $\tilde{D}^{(s)}(a, \beta)$.

$$\begin{aligned}
\mathcal{G}^{(0)} &= \mathbf{I} \\
\mathcal{G}^{(-1)} &= \int \frac{d^2\beta}{\pi} |\beta\rangle\langle \beta| \odot |\beta\rangle\langle \beta|
\end{aligned} \tag{151}$$

$$\begin{aligned}
\mathcal{G}^{(s)}(D(a, \alpha)) &= \int \frac{d^2\gamma}{\pi} e^{s|\gamma|^2} D^\dagger(a, \gamma)D(a, \alpha)D(a, \gamma) \\
&= D(a, \alpha) \int \frac{d^2\gamma}{\pi} e^{s|\gamma|^2} D(\gamma, \alpha) \\
&= D(a, \alpha) \left(-\frac{1}{s} e^{|\alpha|^2/s} \right), \quad s \leq 0
\end{aligned} \tag{152}$$

$$\lim_{s \rightarrow -\infty} [-s\mathcal{G}^{(s)}(D(a, \alpha))] = D(a, \alpha) \implies \lim_{s \rightarrow -\infty} (-s\mathcal{G}^{(s)}) = \mathcal{I} \tag{153}$$

$$1 = -s\mathcal{G}^{(s)}(1) = -s \int \frac{d^2\beta}{\pi} (\tilde{D}^{(s)}(a, \beta))^2, \quad s \leq 0. \tag{154}$$

$$\mathcal{G}^{(s)}|D(a, \alpha) = \int \frac{d^2\beta}{\pi} D^{(s)}(a, \beta)\text{tr}(D^{(s)\dagger}(a, \beta)D(a, \alpha)) = e^{s|\alpha|^2}D(a, \alpha) \tag{155}$$

$$1 = \mathcal{G}^{(s)}|1 = \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) \tag{156}$$

13. Operators and associated functions

$$A = \mathbf{I}|A) = \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) \underbrace{\text{tr}(AD^{(s)}(a, \alpha))}_{\equiv F_A^{(s)}(\alpha)} = \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) F_A^{(s)}(\alpha) \quad (157)$$

$$F_A^{(s)}(\alpha) = F_{A^\dagger}^{(s)*}(-\alpha) = e^{s|\alpha|^2/2} F_A^{(0)}(\alpha) \quad (158)$$

$$F_A^{(s)}(0) = \text{tr}(A) \quad (159)$$

$$F_P^{(s)}(\alpha) = e^{s|\alpha|^2/2} F_P^{(0)}(\alpha) = \frac{1}{2} e^{s|\alpha|^2/2} \quad (160)$$

$$\begin{aligned} \text{tr}(A^\dagger B) &= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \underbrace{\text{tr}(D^{(s)}(a, \alpha) D^{(-s)\dagger}(a, \beta))}_{= \pi \delta^2(\beta - \alpha)} F_A^{(-s)*}(\alpha) F_B^{(s)}(\beta) \\ &= \int \frac{d^2\alpha}{\pi} F_A^{(-s)*}(\alpha) F_B^{(s)}(\alpha) \end{aligned} \quad (161)$$

$$\begin{aligned} A^\dagger B &= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} D(a, \alpha) D^\dagger(a, \beta) F_A^{(0)*}(\alpha) F_B^{(0)}(\beta) \\ &= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} D(\beta, -\alpha/2) D^\dagger(a, \beta - \alpha) F_A^{(0)*}(\alpha) F_B^{(0)}(\beta) \\ &= \int \frac{d^2\mu}{\pi} D^\dagger(a, \mu) \underbrace{\int \frac{d^2\nu}{\pi} D(\nu, \mu/2) F_A^{(0)*}(\nu - \mu/2) F_B^{(0)}(\nu + \mu/2)}_{= F_{A^\dagger B}^{(0)}(\mu)} \end{aligned} \quad (162)$$

$$A = \mathbf{I}|A) = \int \frac{d^2\beta}{\pi} \tilde{D}^{(-s)}(a, \beta) \underbrace{\text{tr}(A \tilde{D}^{(s)}(a, \beta))}_{\equiv \tilde{F}_A^{(s)}(\beta)} = \int \frac{d^2\beta}{\pi} \tilde{D}^{(-s)}(a, \beta) \tilde{F}_A^{(s)}(\beta) \quad (163)$$

$$\tilde{F}_A^{(s)}(\beta) = \tilde{F}_{A^\dagger}^{(s)*}(\beta) \quad (164)$$

$$\text{tr}(A) = \int \frac{d^2\beta}{\pi} \tilde{F}_A^{(s)}(\beta) \quad (165)$$

$$\begin{aligned} \text{tr}(A^\dagger B) &= \int \frac{d^2\beta}{\pi} \frac{d^2\gamma}{\pi} \underbrace{\text{tr}(\tilde{D}^{(s)}(a, \beta) \tilde{D}^{(-s)}(a, \gamma))}_{= \pi \delta^2(\gamma - \beta)} \tilde{F}_A^{(-s)*}(\beta) \tilde{F}_B^{(s)}(\gamma) = \int \frac{d^2\beta}{\pi} \tilde{F}_A^{(-s)*}(\beta) \tilde{F}_B^{(s)}(\beta) \\ &= \pi \delta^2(\gamma - \beta) \end{aligned} \quad (166)$$

$$\begin{aligned} A^\dagger B &= \int \frac{d^2\beta}{\pi} \frac{d^2\gamma}{\pi} \tilde{F}_A^{(0)*}(\beta) \tilde{F}_B^{(0)}(\gamma) \underbrace{\tilde{D}^{(0)}(a, \beta) \tilde{D}^{(0)}(a, \gamma)}_{= 4D^\dagger(a, 2(\gamma - \beta))D(\beta, 2\gamma)} \\ &= \int \frac{d^2\mu}{\pi} D^\dagger(a, \mu) \underbrace{\int \frac{d^2\nu}{\pi} D(\nu, \mu) \tilde{F}_A^{(0)*}(\nu - \mu/4) \tilde{F}_B^{(0)}(\nu + \mu/4)}_{= F_{A^\dagger B}^{(0)}(\mu)} \end{aligned} \quad (167)$$

$$\tilde{F}_A^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi} F_A^{(s)}(\alpha) D(\alpha, \beta) \quad F_A^{(s)}(\alpha) = \int \frac{d^2\beta}{\pi} \tilde{F}_A^{(s)}(\beta) D(\beta, \alpha) \quad (168)$$

$$\begin{aligned}
\tilde{F}_A^{(s)}(\beta) &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} \tilde{F}_A^{(s')}(\gamma) e^{-2|\gamma - \beta|^2/(s' - s)} \\
&= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} \tilde{F}_A^{(s')}(\beta + \gamma) e^{-2|\gamma|^2/(s' - s)}, \quad s \leq s'
\end{aligned} \tag{169}$$

$$\begin{aligned}
F_{A^\dagger B}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi} D(\nu, \mu/2) F_A^{(0)*}(\nu - \mu/2) F_B^{(0)}(\nu + \mu/2) \\
&= \int \frac{d^2\nu}{\pi} D(\nu, \mu) \tilde{F}_A^{(0)*}(\nu - \mu/4) \tilde{F}_B^{(0)}(\nu + \mu/4)
\end{aligned} \tag{170}$$

$$\begin{aligned}
F_{-i[A, B]}^{(0)}(\mu) &= -i \int \frac{d^2\nu}{\pi} [D(\nu, \mu/2) - D^*(\nu, \mu/2)] F_A^{(0)}(\mu/2 - \nu) F_B^{(0)}(\mu/2 + \nu) \\
&= 2 \int \frac{d^2\nu}{\pi} \sin\left(i \frac{\nu\mu^* - \nu^*\mu}{2}\right) F_A^{(0)}(\mu/2 - \nu) F_B^{(0)}(\mu/2 + \nu)
\end{aligned} \tag{171}$$

$$\begin{aligned}
\tilde{F}_{A^\dagger B}^{(0)}(\alpha) &= \int \frac{d^2\mu}{\pi} F_{A^\dagger B}^{(0)}(\mu) D(\mu, \alpha) \\
&= \int \frac{d^2\mu}{\pi} \frac{d^2\nu}{\pi} D(\mu, \alpha - \nu) \tilde{F}_A^{(0)*}(\nu - \mu/4) \tilde{F}_B^{(0)}(\nu + \mu/4) \\
&= 4 \int \frac{d^2\gamma}{\pi} \frac{d^2\delta}{\pi} D(\gamma, 2\delta) \tilde{F}_A^{(0)*}(\alpha + \gamma) \tilde{F}_B^{(0)}(\alpha + \delta)
\end{aligned} \tag{172}$$

$$\begin{aligned}
\tilde{F}_{-i[A, B]}^{(0)}(\alpha) &= -4i \int \frac{d^2\gamma}{\pi} \frac{d^2\delta}{\pi} [D(\gamma, 2\delta) - D^*(\gamma, 2\delta)] \tilde{F}_A^{(0)}(\alpha + \gamma) \tilde{F}_B^{(0)}(\alpha + \delta) \\
&= 8 \int \frac{d^2\gamma}{\pi} \frac{d^2\delta}{\pi} \sin[2i(\gamma\delta^* - \gamma^*\delta)] \tilde{F}_A^{(0)}(\alpha + \gamma) \tilde{F}_B^{(0)}(\alpha + \delta) \\
&= \frac{1}{2} \int \frac{d^2\mu}{\pi} \frac{d^2\nu}{\pi} \sin\left(i \frac{\mu\nu^* - \mu^*\nu}{2}\right) \tilde{F}_A^{(0)}(\alpha + \mu/2) \tilde{F}_B^{(0)}(\alpha + \nu/2)
\end{aligned} \tag{173}$$

$$\begin{aligned}
\tilde{F}_A^{(s)}(\beta) &= \text{tr}(A \tilde{D}^{(s)}(a, \beta)) \\
&= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} \text{tr}(A D(a, \gamma) \tilde{D}^{(s')}(a, \beta) D^\dagger(a, \gamma)) e^{-2|\gamma|^2/(s' - s)} \\
&= \text{tr}\left(\underbrace{\left(\frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} D^\dagger(a, \gamma) A D(a, \gamma) e^{-2|\gamma|^2/(s' - s)}\right)}_{= A'} \tilde{D}^{(s')}(a, \beta)\right) \\
&= \tilde{F}_{A'}^{(s')}(\beta), \quad s \leq s'
\end{aligned} \tag{174}$$

$$A = \sum_{k, l} f_{kl}^{(-s)} [(a^\dagger)^k a^l]_{(-s)} \iff \tilde{F}_A^{(s)}(\beta) = \sum_{k, l} f_{kl}^{(-s)} (\beta^*)^k \beta^l \tag{175}$$

$$s = +1: \begin{cases} A = \int \frac{d^2\beta}{\pi} \tilde{D}^{(-1)}(a, \beta) \tilde{F}_A^{(+1)}(\beta) = \int \frac{d^2\beta}{\pi} |\beta\rangle \langle \beta| \tilde{F}_A^{(+1)}(\beta) \\ \tilde{F}_A^{(+1)}(\beta) = \int \frac{d^2\alpha}{\pi} F_A^{(+1)}(\alpha) D(\alpha, \beta) = \int \frac{d^2\alpha}{\pi} \text{tr}(A e^{\alpha a^\dagger} e^{-\alpha^* a}) D(\alpha, \beta) \end{cases} \tag{176}$$

$$s = 0: \begin{cases} A = \int \frac{d^2\beta}{\pi} \tilde{D}^{(0)}(a, \beta) \tilde{F}_A^{(0)}(\beta) = 2P \int \frac{d^2\beta}{\pi} D(a, -2\beta) \tilde{F}_A^{(0)}(\beta) \\ \tilde{F}_A^{(0)}(\beta) = \int \frac{d^2\alpha}{\pi} F_A^{(0)}(\alpha) D(\alpha, \beta) = \int \frac{d^2\alpha}{\pi} \text{tr}(A D(a, \alpha)) D(\alpha, \beta) \\ \tilde{F}_A^{(0)}(\beta) = \text{tr}(A \tilde{D}^{(0)}(a, \beta)) = 2 \text{tr}(A P D(a, -2\beta)) = 2 \int \frac{d^2\alpha}{\pi} \langle \beta - \alpha | A | \beta + \alpha \rangle D(\alpha, \beta) \end{cases} \tag{177}$$

$$\tilde{F}_P^{(0)}(\beta) = 2 \operatorname{tr}(D(a, -2\beta)) = 2\pi\delta^2(-2\beta) = \frac{\pi}{2}\delta^2(\beta) \quad (178)$$

$$\begin{aligned} \langle x|A|x' \rangle &= \int \frac{d\beta_1 d\beta_2}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}(\beta_1 + i\beta_2)\right) \langle x|2PD(a, -2\beta)|x' \rangle \\ &= \int \frac{d\beta_1 d\beta_2}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}(\beta_1 + i\beta_2)\right) e^{i\beta_2(x-x')} \delta(\beta_1 - (x+x')/2) \\ &= \int \frac{d\beta_2}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}((x+x')/2 + i\beta_2)\right) e^{i\beta_2(x-x')} \\ \tilde{F}_A^{(0)}(\beta) &= 2 \int dx \langle x|APD(a, -2\beta)|x \rangle \\ &= \int dx dx' \langle x|A|x' \rangle \langle x'|2PD(a, -2\beta)|x \rangle \\ &= \int dx dx' \langle x|A|x' \rangle e^{-i\beta_2(x-x')} \delta(\beta_1 - (x+x')/2) \\ &= \int dX d\xi \langle X + \xi/2|A|X - \xi/2 \rangle e^{-i\beta_2\xi} \delta(\beta_1 - X) \\ &= \int d\xi \langle \beta_1 + \xi/2|A|\beta_1 - \xi/2 \rangle e^{-i\beta_2\xi} \end{aligned} \quad (179)$$

$$\begin{aligned} \langle p|A|p' \rangle &= \int \frac{d\beta_1 d\beta_2}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}(\beta_1 + i\beta_2)\right) \langle p|2PD(a, -2\beta)|p' \rangle \\ &= \int \frac{d\beta_1 d\beta_2}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}(\beta_1 + i\beta_2)\right) e^{-i\beta_1(p-p')} \delta(\beta_2 - (p+p')/2) \\ &= \int \frac{d\beta_1}{2\pi} \tilde{F}_A^{(0)}\left(\frac{1}{\sqrt{2}}(\beta_1 + i(p+p')/2)\right) e^{-i\beta_1(p-p')} \\ \tilde{F}_A^{(0)}(\beta) &= 2 \int dp \langle p|APD(a, -2\beta)|p \rangle \\ &= \int dp dp' \langle p|A|p' \rangle \langle p'|2PD(a, -2\beta)|p \rangle \\ &= \int dp dp' \langle p|A|p' \rangle e^{i\beta_1(p-p')} \delta(\beta_2 - (p+p')/2) \\ &= \int dP d\eta \langle P + \eta/2|A|P - \eta/2 \rangle e^{i\beta_1\eta} \delta(\beta_2 - P) \\ &= \int d\eta \langle \beta_2 + \eta/2|A|\beta_2 - \eta/2 \rangle e^{i\beta_1\eta} \end{aligned} \quad (180)$$

$$s = -1: \begin{cases} A = \int \frac{d^2\beta}{\pi} \tilde{D}^{(+1)}(a, \beta) \tilde{F}_A^{(-1)}(\beta) \\ \tilde{F}_A^{(-1)}(\beta) = \operatorname{tr}(A\tilde{D}^{(-1)}(a, \beta)) = \langle \beta|A|\beta \rangle \end{cases} \quad (181)$$

$$\tilde{F}_P^{(-1)}(\beta) = \langle \beta|P|\beta \rangle = \langle \beta|-\beta \rangle = e^{-2|\beta|^2} \quad (182)$$

$$\begin{aligned}
F_{aA}^{(s)}(\alpha) &= e^{(s-1)|\alpha|^2/2} F_{aA}^{(+1)}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial(-\alpha^*)} F_A^{(+1)}(\alpha) \\
&= -e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha^*} e^{-(s-1)|\alpha|^2/2} F_A^{(s)}(\alpha) \\
&= \left(-\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha \right) F_A^{(s)}(\alpha) \\
F_{a^\dagger A}^{(s)}(\alpha) &= e^{(s+1)|\alpha|^2/2} F_{a^\dagger A}^{(-1)}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} F_A^{(-1)}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} e^{-(s+1)|\alpha|^2/2} F_A^{(s)}(\alpha) \\
&= \left(\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^* \right) F_A^{(s)}(\alpha) \\
F_{Aa^\dagger}^{(s)}(\alpha) &= e^{(s-1)|\alpha|^2/2} F_{Aa^\dagger}^{(+1)}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} F_A^{(+1)}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} e^{-(s-1)|\alpha|^2/2} F_A^{(s)}(\alpha) \\
&= \left(\frac{\partial}{\partial\alpha} + \frac{1-s}{2}\alpha^* \right) F_A^{(s)}(\alpha) \\
F_{Aa}^{(s)}(\alpha) &= e^{(s+1)|\alpha|^2/2} F_{Aa}^{(-1)}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial(-\alpha^*)} F_A^{(-1)}(\alpha) \\
&= -e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha^*} e^{-(s+1)|\alpha|^2/2} F_A^{(s)}(\alpha) \\
&= \left(-\frac{\partial}{\partial\alpha^*} + \frac{1+s}{2}\alpha \right) F_A^{(s)}(\alpha)
\end{aligned} \tag{183}$$

$$\begin{aligned}
\tilde{F}_{aA}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi} F_{aA}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} \left[\left(-\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha \right) F_A^{(s)}(\alpha) \right] D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} F_A^{(s)}(\alpha) \left(\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha \right) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} F_A^{(s)}(\alpha) \left(\beta + \frac{1-s}{2} \frac{\partial}{\partial\beta^*} \right) D(\alpha, \beta) \\
&= \left(\beta + \frac{1-s}{2} \frac{\partial}{\partial\beta^*} \right) \tilde{F}_A^{(s)}(\beta) \\
\tilde{F}_{a^\dagger A}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi} F_{a^\dagger A}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} \left[\left(\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^* \right) F_A^{(s)}(\alpha) \right] D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} F_A^{(s)}(\alpha) \left(-\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^* \right) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi} F_A^{(s)}(\alpha) \left(\beta^* - \frac{1+s}{2} \frac{\partial}{\partial\beta} \right) D(\alpha, \beta) \\
&= \left(\beta^* - \frac{1+s}{2} \frac{\partial}{\partial\beta} \right) \tilde{F}_A^{(s)}(\beta) \\
\tilde{F}_{Aa^\dagger}^{(s)}(\beta) &= \tilde{F}_{aA^\dagger}^{(s)*}(\beta) = \left(\beta^* + \frac{1-s}{2} \frac{\partial}{\partial\beta} \right) \tilde{F}_A^{(s)}(\beta) \\
\tilde{F}_{Aa}^{(s)}(\beta) &= \tilde{F}_{a^\dagger A^\dagger}^{(s)*}(\beta) = \left(\beta - \frac{1+s}{2} \frac{\partial}{\partial\beta^*} \right) \tilde{F}_A^{(s)}(\beta)
\end{aligned} \tag{184}$$

$$\begin{aligned}
\mathcal{G}^{(0)} &= \mathbf{I} \\
\mathcal{G}^{(-1)} &= \int \frac{d^2\beta}{\pi} |\beta\rangle\langle\beta| \odot |\beta\rangle\langle\beta|
\end{aligned} \tag{185}$$

$$\begin{aligned}
\mathcal{G}^{(s)}|A\rangle &= \int \frac{d^2\alpha}{\pi} D^{(s)\dagger}(a, \alpha) \text{tr}(D^{(s)}(a, \alpha)A) = \int \frac{d^2\alpha}{\pi} D^{(s)\dagger}(a, \alpha) F_A^{(s)}(\alpha) \\
&= \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) \text{tr}(\tilde{D}^{(s)}(a, \beta)A) = \int \frac{d^2\beta}{\pi} \tilde{D}^{(s)}(a, \beta) \tilde{F}_A^{(s)}(\beta)
\end{aligned} \tag{186}$$

$$\mathcal{G}^{(s)}(D^{(s')}(a, \alpha)) = D^{(s')}(a, \alpha) \left(-\frac{1}{s} e^{|\alpha|^2/s} \right) = -\frac{1}{s} D(a, \alpha) e^{|\alpha|^2(1/s+s'/2)} = -\frac{1}{s} D^{(s'+2/s)}(a, \alpha), \quad s \leq 0 \tag{187}$$

$$\begin{aligned}
\mathcal{G}^{(s)}(A) &= -\frac{1}{s} \int \frac{d^2\alpha}{\pi} D^{(2/s-s')\dagger}(a, \alpha) F_A^{(s')}(\alpha) \\
&= -\frac{1}{s} \int \frac{d^2\beta}{\pi} \tilde{D}^{(2/s-s')}(a, \beta) \tilde{F}_A^{(s')}(\beta) \\
&= -\frac{1}{s} \int \frac{d^2\beta}{\pi} D(a, \beta) \tilde{D}^{(2/s-s')}(a, 0) D^\dagger(a, \beta) \tilde{F}_A^{(s')}(\beta) \\
&= \int \frac{d^2\beta}{\pi} D(a, \beta) (s+1)^{a^\dagger a} D^\dagger(a, \beta) \tilde{F}_A^{(-1)}(\beta)
\end{aligned} \tag{188}$$

$$\mathcal{G}^{(s)}(\tilde{D}^{(s')}(a, \beta)) = -\frac{1}{s} \tilde{D}^{(s'+2/s)}(a, \beta) \tag{189}$$

$$\mathcal{G}^{(s)}\left(\left(\frac{s'+1}{s'-1}\right)^{a^\dagger a}\right) = -\frac{1}{s}\frac{1-s'}{1-s'-2/s}\left(\frac{s'+2/s+1}{s'+2/s-1}\right)^{a^\dagger a} \quad (190)$$

14. Characteristic functions and quasiprobability distributions

$$\left(\begin{array}{c} s\text{-ordered} \\ \text{characteristic} \\ \text{function} \end{array} \right) = \Phi_\rho^{(s)}(\alpha) \equiv F_\rho^{(s)}(\alpha) = \text{tr}(\rho D^{(s)}(a, \alpha)) = \Phi_\rho^{(s)*}(-\alpha) = e^{s|\alpha|^2/2} \Phi_\rho^{(0)}(\alpha) \quad (191)$$

$$\rho = \int \frac{d^2\alpha}{\pi} D^{(-s)\dagger}(a, \alpha) \Phi_\rho^{(s)}(\alpha) \quad (192)$$

$$\Phi_\rho^{(s)}(\alpha) = \sum_{k,l} \frac{\alpha^k (-\alpha^*)^l}{k! l!} \text{tr}(\rho [(a^\dagger)^k a^l]_{(s)}) \quad (193)$$

$$\text{tr}(\rho [(a^\dagger)^k a^l]_{(s)}) = \left. \frac{\partial^{k+l} \Phi_\rho^{(s)}(\alpha)}{\partial \alpha^k \partial (-\alpha^*)^l} \right|_{\alpha=0} \quad (194)$$

$$\Phi_\rho^{(s)}(0) = \text{tr}(\rho) = 1 \quad (195)$$

$$\text{tr}(\rho_1 \rho_2) = \int \frac{d^2\alpha}{\pi} \Phi_{\rho_1}^{(-s)*}(\alpha) \Phi_{\rho_2}^{(s)}(\alpha) \quad (196)$$

$$\text{tr}(\rho^2) = \int \frac{d^2\alpha}{\pi} \Phi_\rho^{(-s)*}(\alpha) \Phi_\rho^{(s)}(\alpha) \quad (197)$$

$$|\Phi_\rho^{(s)}(\alpha)| \leq e^{s|\alpha|^2/2} \quad (198)$$

$$\Phi_\rho^{(s)}(\alpha) = e^{(s-s')|\alpha|^2/2} \Phi_\rho^{(s')}(\alpha) \quad (199)$$

$$\left(\begin{array}{c} s\text{-ordered} \\ \text{quasiprobability} \\ \text{distribution} \end{array} \right) = W_\rho^{(s)}(\beta) \equiv \frac{1}{\pi} \tilde{F}_\rho^{(s)}(\beta) = \frac{1}{\pi} \text{tr}(\rho \tilde{D}^{(s)}(a, \beta)) = W_\rho^{(s)*}(\beta) \quad (200)$$

$$\rho = \int d^2\beta \tilde{D}^{(-s)}(a, \beta) W_\rho^{(s)}(\beta) \quad (201)$$

$$\text{tr}(\rho [(a^\dagger)^k a^l]_{(s)}) = \int d^2\beta \text{tr}(\tilde{D}^{(-s)}(a, \beta) [(a^\dagger)^k a^l]_{(s)}) W_\rho^{(s)}(\beta) = \int d^2\beta (\beta^*)^k \beta^l W_\rho^{(s)}(\beta) \quad (202)$$

$$\text{tr}(\rho A) = \int \frac{d^2\beta}{\pi} \tilde{F}_\rho^{(-s)*}(\beta) \tilde{F}_A^{(s)}(\beta) = \int d^2\beta \tilde{F}_A^{(s)}(\beta) W_\rho^{(-s)}(\beta) \quad (203)$$

$$1 = \text{tr}(\rho) = \int d^2\beta W_\rho^{(s)}(\beta) \quad (204)$$

$$\text{tr}(\rho_1 \rho_2) = \pi \int d^2\beta W_{\rho_1}^{(-s)}(\beta) W_{\rho_2}^{(s)}(\beta) \quad (205)$$

$$\text{tr}(\rho^2) = \pi \int d^2\beta W_\rho^{(-s)}(\beta) W_\rho^{(s)}(\beta) \quad (206)$$

$$W_\rho^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi^2} \Phi_\rho^{(s)}(\alpha) D(\alpha, \beta) \quad \Phi_\rho^{(s)}(\alpha) = \int d^2\beta W_\rho^{(s)}(\beta) D(\beta, \alpha) \quad (207)$$

$$\begin{aligned} W_\rho^{(s)}(\beta) &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} W_\rho^{(s')}(\gamma) e^{-2|\gamma - \beta|^2/(s' - s)} \\ &= \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} W_\rho^{(s')}(\beta + \gamma) e^{-2|\gamma|^2/(s' - s)}, \quad s \leq s' \end{aligned} \quad (208)$$

$$W_\rho^{(s)}(\beta) = W_{\rho'}^{(s')}(\beta), \text{ for } s \leq s', \text{ where}$$

$$\rho' = \frac{2}{s' - s} \int \frac{d^2\gamma}{\pi} D^\dagger(a, \gamma) \rho D(a, \gamma) e^{-2|\gamma|^2/(s' - s)} \quad (209)$$

$$F_{\rho^2}^{(0)}(\mu) = \int \frac{d^2\nu}{\pi} D(\nu, \mu/2) \Phi_\rho^{(-s)*}(\nu - \mu/2) \Phi_\rho^{(s)}(\nu + \mu/2)$$

$$= \pi \int d^2\nu D(\nu, \mu) W_\rho^{(-s)}(\nu - \mu/4) W_\rho^{(s)}(\nu + \mu/4) \quad (210)$$

$$\Phi_{|\psi\rangle\langle\psi|}^{(0)}(\mu) = \int \frac{d^2\nu}{\pi} D(\nu, \mu/2) \Phi_{|\psi\rangle\langle\psi|}^{(-s)*}(\nu - \mu/2) \Phi_{|\psi\rangle\langle\psi|}^{(s)}(\nu + \mu/2)$$

$$= \pi \int d^2\nu D(\nu, \mu) W_{|\psi\rangle\langle\psi|}^{(-s)}(\nu - \mu/4) W_{|\psi\rangle\langle\psi|}^{(s)}(\nu + \mu/4) \quad (211)$$

$$s = +1: \left\{ \begin{array}{l} \frac{1}{\pi} \tilde{F}_\rho^{(+1)}(\beta) = W_\rho^{(+1)}(\beta) \equiv P(\beta) = \left(\begin{array}{c} \text{Glauber} \\ P \text{ function} \end{array} \right) \\ \rho = \int d^2\beta \tilde{D}^{(-1)}(a, \beta) P(\beta) = \int d^2\beta P(\beta) |\beta\rangle\langle\beta| \\ P(\beta) = \int \frac{d^2\alpha}{\pi^2} \Phi_\rho^{(+1)}(\alpha) D(\alpha, \beta) = \int \frac{d^2\alpha}{\pi^2} \text{tr}(\rho e^{\alpha\alpha^\dagger} e^{-\alpha^* a}) D(\alpha, \beta) \end{array} \right. \quad (212)$$

$$s = 0: \left\{ \begin{array}{l} \frac{1}{\pi} \tilde{F}_\rho^{(0)}(\beta) = W_\rho^{(0)}(\beta) \equiv W(\beta) = \left(\begin{array}{c} \text{Wigner} \\ \text{function} \end{array} \right) \\ \rho = \int d^2\beta \tilde{D}^{(0)}(a, \beta) W(\beta) = 2P \int d^2\beta D(a, -2\beta) W(\beta) \\ W(\beta) = \int \frac{d^2\alpha}{\pi^2} \Phi_\rho^{(0)}(\alpha) D(\alpha, \beta) = \int \frac{d^2\alpha}{\pi^2} \text{tr}(\rho D(a, \alpha)) D(\alpha, \beta) \\ W(\beta) = \frac{1}{\pi} \text{tr}(\rho \tilde{D}^{(0)}(a, \beta)) = \underbrace{\frac{2}{\pi} \text{tr}(\rho D(a, \beta) P D^\dagger(a, \beta))}_{\leq 2/\pi} = \frac{2}{\pi} \text{tr}(\rho P D(a, -2\beta)) \\ = \frac{2}{\pi^2} \int d^2\alpha \langle \beta - \alpha | \rho | \beta + \alpha \rangle D(\alpha, \beta) \\ \text{tr}(\rho_1 \rho_2) = \pi \int d^2\beta W_{\rho_1}(\beta) W_{\rho_2}(\beta) \quad \text{tr}(\rho^2) = \pi \int d^2\beta W^2(\beta) \end{array} \right. \quad (213)$$

If position β_1 and momentum β_2 are used as the variables in the Wigner function, it is conventional to use a rescaled Wigner function defined by

$$W'(\beta_1, \beta_2) = \frac{1}{2} W(\beta) = \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} \text{tr}(\rho e^{i(\alpha_2 x - \alpha_1 p)}) e^{i(\beta_2 \alpha_1 - \beta_1 \alpha_2)}. \quad (214)$$

$$\langle x | \rho | x' \rangle = \int \frac{d\beta_2}{2} W\left(\frac{1}{\sqrt{2}}\left((x + x')/2 + i\beta_2\right)\right) e^{i\beta_2(x - x')}$$

$$= \int d\beta_2 W'((x + x')/2, \beta_2) e^{i\beta_2(x - x')} \quad (215)$$

$$W'(\beta_1, \beta_2) = \frac{1}{2} W(\beta) = \int \frac{d\xi}{2\pi} \langle \beta_1 + \xi/2 | \rho | \beta_1 - \xi/2 \rangle e^{-i\beta_2 \xi}$$

$$\begin{aligned}
\langle p|\rho|p'\rangle &= \int \frac{d\beta_1}{2} W\left(\frac{1}{\sqrt{2}}\left(\beta_1 + i(p+p')/2\right)\right) e^{-i\beta_1(p-p')} \\
&= \int d\beta_1 W'(\beta_1, (p+p')/2) e^{-i\beta_1(p-p')}
\end{aligned} \tag{216}$$

$$\begin{aligned}
W'(\beta_1, \beta_2) &= \frac{1}{2} W(\beta) = \int \frac{d\eta}{2\pi} \langle \beta_2 + \eta/2 | \rho | \beta_2 - \eta/2 \rangle e^{i\beta_1 \eta} \\
1 = \text{tr}(\rho) &= \int d^2\beta W(\beta) = \int d\beta_1 d\beta_2 W'(\beta_1, \beta_2)
\end{aligned} \tag{217}$$

$$\text{tr}(\rho_1 \rho_2) = 2\pi \int d\beta_1 d\beta_2 W'_{\rho_1}(\beta_1, \beta_2) W'_{\rho_2}(\beta_1, \beta_2) \quad \text{tr}(\rho^2) = 2\pi \int d\beta_1 d\beta_2 W'^2(\beta_1, \beta_2) \tag{218}$$

$$s = -1: \begin{cases} \frac{1}{\pi} \tilde{F}_\rho^{(-1)}(\beta) = W_\rho^{(-1)}(\beta) \equiv Q(\beta) = \left(\begin{array}{c} \text{Husimi} \\ Q \text{ function} \end{array} \right) \\ \rho = \int d^2\beta \tilde{D}^{(+1)}(a, \beta) Q(\beta) \\ Q(\beta) = \frac{1}{\pi} \text{tr}(\rho \tilde{D}^{(-1)}(a, \beta)) = \frac{1}{\pi} \langle \beta | \rho | \beta \rangle \leq \frac{1}{\pi} \end{cases} \tag{219}$$

$$\Phi_{|\gamma\rangle\langle\gamma|}^{(s)}(\alpha) = \langle \gamma | D^{(s)}(a, \alpha) | \gamma \rangle = e^{(s-1)|\alpha|^2/2} D(\gamma, \alpha) \tag{220}$$

$$\begin{aligned}
W_{|\gamma\rangle\langle\gamma|}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2} \Phi_{|\gamma\rangle\langle\gamma|}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi^2} e^{(s-1)|\alpha|^2/2} D(\alpha, \beta - \gamma) \\
&= \frac{2}{\pi(1-s)} e^{-2|\beta-\gamma|^2/(1-s)}
\end{aligned} \tag{221}$$

$$\rho = |\gamma\rangle\langle\gamma|: \begin{cases} s = +1: & P(\beta) = \delta^2(\beta - \gamma) \\ s = 0: & W(\beta) = \frac{2}{\pi} e^{-2|\beta-\gamma|^2}, \quad W'(\beta_1, \beta_2) = \frac{1}{\pi} e^{-(\beta_1-\gamma_1)^2 - (\beta_2-\gamma_2)^2} \\ s = -1: & Q(\beta) = \frac{1}{\pi} e^{-|\beta-\gamma|^2} = \frac{1}{\pi} |\langle \beta | \gamma \rangle|^2 \end{cases} \tag{222}$$

$$\begin{aligned}
\rho = |\gamma\rangle\langle\gamma|: \quad W(\beta) &= \frac{2}{\pi} \langle \gamma | D^\dagger(a, -\beta) P D(a, -\beta) | \gamma \rangle \\
&= \frac{2}{\pi} \langle \gamma - \beta | P | \gamma - \beta \rangle \\
&= \frac{2}{\pi} \langle \gamma - \beta | \beta - \gamma \rangle \\
&= \frac{2}{\pi} e^{-2|\beta-\gamma|^2}
\end{aligned} \tag{223}$$

The remainder of this section deals with number-state projectors, i.e., $\rho = |n\rangle\langle n|$.

$$\Phi^{(s)}(\alpha) = e^{s|\alpha|^2/2} \langle n | D(a, \alpha) | n \rangle = e^{(s-1)|\alpha|^2/2} L_n(|\alpha|^2) \tag{224}$$

$$\begin{aligned}
W^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2} \Phi^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi^2} e^{(s-1)|\alpha|^2/2} L_n(|\alpha|^2) D(\alpha, \beta) \\
(\beta = \beta^*) &= \frac{1}{\pi} \int_0^\infty dx e^{-(1-s)x/2} L_n(x) \frac{1}{2\pi} \int_{-\pi}^\pi d\phi e^{-2i\beta\sqrt{x}\sin\phi} \\
&= \frac{1}{\pi} \int_0^\infty dx e^{-(1-s)x/2} L_n(x) \underbrace{\frac{1}{\pi} \int_0^\pi d\phi \cos(2\beta\sqrt{x}\sin\phi)}_{= \pi J_0(2\beta\sqrt{x})} \\
(\beta \text{ general}) &= \frac{1}{\pi} \int_0^\infty dx e^{-(1-s)x/2} L_n(x) J_0(2|\beta|\sqrt{x})
\end{aligned} \tag{225}$$

$$\begin{aligned}
W^{(s)}(\beta) &= \frac{1}{\pi} \int_0^\infty dx e^{-(1-s)x/2} L_n(x) J_0(2|\beta|\sqrt{x}) \\
&= \frac{1}{\pi} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(-1)^k}{k!} \int_0^\infty dx e^{-(1-s)x/2} x^k J_0(2|\beta|\sqrt{x}) \\
&= \frac{2}{\pi(1-s)} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{1}{k!} \left(\frac{2}{s-1}\right)^k \underbrace{\int_0^\infty dt e^{-t} t^k J_0\left(2|\beta|\sqrt{\frac{2}{1-s}t}\right)}_{= k! e^{-2|\beta|^2/(1-s)} L_k\left(\frac{2}{1-s}|\beta|^2\right)} \\
&= \frac{2}{\pi(1-s)} e^{-2|\beta|^2/(1-s)} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \left(\frac{2}{s-1}\right)^k L_k\left(\frac{2}{1-s}|\beta|^2\right) \\
&= \frac{2}{\pi(1-s)} \left(\frac{s+1}{s-1}\right)^n e^{-2|\beta|^2/(1-s)} L_n\left(\frac{4}{1-s^2}|\beta|^2\right)
\end{aligned} \tag{226}$$

Equation (226) uses

$$k! e^{-x} L_k(x) = \int_0^\infty dt e^{-t} t^k J_0(2\sqrt{tx}) \quad (\text{A\&S 22.10.14})$$

and

$$\sum_{k=0}^n \frac{n!}{k!(n-k)!} \mu^k L_k(x) = (1+\mu)^n L_n\left(\frac{\mu}{1+\mu}x\right) \quad (\text{A\&S 22.12.7})$$

$$\begin{aligned}
s = 0: \quad W(\beta) &= \frac{2}{\pi} \langle n | PD(a, -2\beta) | n \rangle = \frac{2}{\pi} (-1)^n e^{-2|\beta|^2} L_n(4|\beta|^2) \\
W'(\beta_1, \beta_2) &= \frac{(-1)^n}{\pi} e^{-(\beta_1^2 + \beta_2^2)} L_n(2(\beta_1^2 + \beta_2^2))
\end{aligned} \tag{227}$$

$$\begin{aligned}
(-1)^n e^{-|\alpha|^2} L_n(2|\alpha|^2) &= \int \frac{d^2\beta}{\pi} e^{-|\beta|^2} L_n(2|\beta|^2) D(\beta, \alpha) \\
s = -1: \quad Q(\beta) &= \frac{1}{\pi} |\langle \beta | n \rangle|^2 = \frac{1}{\pi} e^{-|\beta|^2} \frac{|\beta|^{2n}}{n!} \\
e^{-|\beta|^2} \frac{|\beta|^{2n}}{n!} &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} L_n(|\alpha|^2) D(\alpha, \beta) \\
e^{-|\alpha|^2} L_n(|\alpha|^2) &= \int \frac{d^2\beta}{\pi} e^{-|\beta|^2} \frac{|\beta|^{2n}}{n!} D(\beta, \alpha)
\end{aligned} \tag{228}$$

$$\begin{aligned}
\langle n | [(a^\dagger)^k a^k]_{(s)} | n \rangle &= \left. \frac{\partial^{2k} \Phi^{(s)}(\alpha)}{\partial \alpha^k \partial (-\alpha^*)^k} \right|_{\alpha=0} = (-1)^k k! \left. \frac{d^k}{dx^k} e^{(s-1)x/2} L_n(x) \right|_{x=0} \\
&= \int d^2 \beta |\beta|^{2k} W^{(s)}(\beta) = \pi \int_0^\infty dx x^k W^{(s)}(x) \\
W^{(s)}(\beta) &\rightarrow W^{(s)}(|\beta|^2) = W^{(s)}(x)
\end{aligned} \tag{229}$$

$$\begin{aligned}
(-1)^k k! L_n^{(k)}(0) &= \langle n | [(a^\dagger)^k a^k]_{(+1)} | n \rangle = \langle n | (a^\dagger)^k a^k | n \rangle = (-1)^k (-n)_k = \begin{cases} n!/(n-k)!, & k \leq n \\ 0, & k > n \end{cases} \\
L_n(x) &= \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(-x)^k}{k!} = {}_1F_1(-n; 1; x) = M(-n, 1, x)
\end{aligned} \tag{230}$$

$$\begin{aligned}
\langle n | [(a^\dagger)^k a^k]_{(-1)} | n \rangle &= \langle n | a^k (a^\dagger)^k | n \rangle \\
&= (-1)^k k! \left. \frac{d^k}{dx^k} e^{-x} L_n(x) \right|_{x=0} \\
&= \frac{1}{n!} \int_0^\infty dx x^{n+k} e^{-x} \\
&= \frac{(n+k)!}{n!} = (n+1)_k
\end{aligned} \tag{231}$$

$$L_n(x) = e^x \sum_{k=0}^\infty \frac{(n+k)!}{n! k!} \frac{(-x)^k}{k!}$$

$$\begin{aligned}
\langle n | [(a^\dagger)^k a^k]_{(s)} | n \rangle &= (-1)^k k! \left. \frac{d^k}{dx^k} e^{(s-1)x/2} L_n(x) \right|_{x=0} \\
&= \left(\frac{1-s}{2} \right)^k k! \underbrace{\sum_{m=0}^{\min(n,k)} \frac{k!}{m!(k-m)!} \frac{n!}{m!(n-m)!} \left(\frac{2}{1-s} \right)^m}_{= {}_2F_1\left(-n, -k; 1; \frac{2}{1-s}\right)} \\
&= \left(\frac{1-s}{2} \right)^k k! {}_2F_1\left(-n, -k; 1; \frac{2}{1-s}\right) \\
&= \left(\frac{1-s}{2} \right)^k \left(\frac{s+1}{s-1} \right)^n k! {}_2F_1\left(-n, k+1; 1; \frac{2}{1+s}\right) \quad (\text{A\&S 15.3.4})
\end{aligned} \tag{232}$$

$$\begin{aligned}
\langle n | [(a^\dagger)^k a^k]_{(0)} | n \rangle &= 2(-1)^n \int_0^\infty dx x^k e^{-2x} L_n(4x) \\
&= \frac{(-1)^n}{2^k} \int_0^\infty du u^k e^{-u} L_n(2u) \\
&= \frac{(-1)^n}{2^k} \underbrace{\sum_{m=0}^n \frac{(m+k)!}{m!} \frac{n!}{m!(n-m)!} (-2)^m}_{= k! {}_2F_1(-n, k+1; 1; 2)} \\
&= \frac{k!}{2^k} (-1)^n {}_2F_1(-n, k+1; 1; 2)
\end{aligned} \tag{233}$$

The sums in Eqs. (232) and (233) are related by

$$\begin{aligned}
\sum_{m=0}^n \frac{(m+k)!}{m!} \frac{n!}{m!(n-m)!} x^m &= \frac{d^k}{dx^k} \sum_{m=0}^n \frac{n!}{m!(n-m)!} x^{m+k} \\
&= \frac{d^k}{dx^k} x^k (1+x)^n \\
&= \sum_{m=0}^{\min(n,k)} \frac{k!}{m!(k-m)!} \frac{d^{k-m} x^k}{dx^{k-m}} \frac{d^m (1+x)^n}{dx^m} \\
&= k! \sum_{m=0}^{\min(n,k)} \frac{k!}{m!(k-m)!} \frac{n!}{m!(n-m)!} x^m (1+x)^{n-m}
\end{aligned} \tag{234}$$

$$\begin{aligned}
\frac{1}{k!} \int_0^\infty du u^k e^{-u} L_n(cu) &= \sum_{m=0}^n \frac{(k+m)!}{k! m!} \frac{n!}{m!(n-m)!} (-c)^m = {}_2F_1(-n, k+1; 1; c) \\
&= (1-c)^n \sum_{m=0}^{\min(n,k)} \frac{k!}{m!(k-m)!} \frac{n!}{m!(n-m)!} \left(\frac{c}{c-1} \right)^m
\end{aligned} \tag{235}$$

$$\begin{aligned}
[(a^\dagger)^k a^k]_{(s)} &= \left(\frac{1-s}{2} \right)^k \left(\frac{s+1}{s-1} \right)^{a^\dagger a} k! {}_2F_1 \left(-a^\dagger a, k+1; 1; \frac{2}{1+s} \right) \\
&= \sum_{m=0}^k \frac{1}{(k-m)!} \left(\frac{1-s}{2} \right)^{k-m} \left(\frac{k!}{m!} \right)^2 (a^\dagger a)^{(m)} \\
&= \sum_{m=0}^k \frac{1}{m!} \left(\frac{1-s}{2} \right)^m \left(\frac{k!}{(k-m)!} \right)^2 (a^\dagger a)^{k-m}
\end{aligned} \tag{236}$$

$$\int \frac{d^2 \alpha}{\pi} e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2) D(\alpha, \beta) = \frac{2}{1+s} \left(\frac{1-s}{1+s} \right)^n e^{-2|\beta|^2/(1+s)} (-1)^n L_n \left(\frac{4}{1-s^2} |\beta|^2 \right) \tag{237}$$

$$\begin{aligned}
\int \frac{d^2 \alpha}{\pi} (-\alpha)^k (\alpha^*)^l e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2) &= \delta_{kl} (-1)^k \underbrace{\int \frac{d^2 \alpha}{\pi} |\alpha|^{2k} e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2)}_{= \int_0^\infty dx x^k e^{-(1+s)x/2} L_n(x)} \\
&= \delta_{kl} (-1)^k \left(\frac{2}{1+s} \right)^{k+1} \underbrace{\int_0^\infty du u^k e^{-u} L_n \left(\frac{2}{1+s} u \right)}_{= k! {}_2F_1(-n, k+1; 1; 2/(1+s))} \\
&= \delta_{kl} (-1)^k k! \left(\frac{2}{1+s} \right)^{k+1} {}_2F_1(-n, k+1; 1; 2/(1+s))
\end{aligned} \tag{238}$$

$$\begin{aligned}
& \int \frac{d^2\alpha}{\pi} e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2) D(\alpha, \beta) \\
&= \sum_{k,l} \frac{(\beta^*)^k \beta^l}{k! l!} \int \frac{d^2\alpha}{\pi} (-\alpha)^k (\alpha^*)^l e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2) \\
&= \sum_{k=0}^{\infty} \frac{(-|\beta|^2)^k}{k!} \left(\frac{2}{1+s} \right)^{k+1} \underbrace{{}_2F_1(-n, k+1; 1; 2/(1+s))}_{= \left(\frac{s-1}{s+1} \right)^n \left(\frac{2}{1-s} \right)^k (-1)^k \frac{d^k}{dx^k} e^{-(1-s)x/2} L_n(x) \Big|_{x=0}} \\
&= (-1)^n \frac{2}{1+s} \left(\frac{1-s}{1+s} \right)^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{4}{1-s^2} |\beta|^2 \right)^k \frac{d^k}{dx^k} e^{-(1-s)x/2} L_n(x) \Big|_{x=0} \\
&= \frac{2}{1+s} \left(\frac{1-s}{1+s} \right)^n e^{-2|\beta|^2/(1+s)} (-1)^n L_n \left(\frac{4}{1-s^2} |\beta|^2 \right)
\end{aligned} \tag{239}$$

$$\begin{aligned}
|n\rangle \langle n| &= \int \frac{d^2\alpha}{\pi} D^{(s)\dagger}(a, \alpha) \langle n | D^{(-s)}(a, \alpha) | n \rangle \\
&= \sum_{k,l} \frac{[(a^\dagger)^k a^l]_{(s)}}{k! l!} \int \frac{d^2\alpha}{\pi} (-\alpha)^k (\alpha^*)^l e^{-(1+s)|\alpha|^2/2} L_n(|\alpha|^2) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{2}{1+s} \right)^{k+1} {}_2F_1 \left(-n, k+1; 1; \frac{2}{1+s} \right) [(a^\dagger)^k a^k]_{(s)}
\end{aligned} \tag{240}$$

15. Thermal states

$$\rho = \frac{1}{Z} e^{-\mu a^\dagger a} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\mu n} |n\rangle \langle n|, \quad \mu = \beta \hbar \omega = \hbar \omega / kT \quad (241)$$

$$Z = \text{tr}(e^{-\mu a^\dagger a}) = \sum_{n=0}^{\infty} e^{-\mu n} = \frac{1}{1 - e^{-\mu}} \quad (242)$$

$$\bar{n} = \langle a^\dagger a \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-\mu n} = -\frac{1}{Z} \frac{\partial Z}{\partial \mu} = \frac{1}{e^\mu - 1} \quad (243)$$

$$\frac{d\bar{n}}{d(-\mu)} = \bar{n}(\bar{n} + 1) \quad (244)$$

$$e^\mu = \frac{1 + \bar{n}}{\bar{n}} = \frac{Z}{Z - 1}$$

$$Z = 1 + \bar{n} = \frac{1}{1 - e^{-\mu}} \quad (245)$$

$$\rho = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^{a^\dagger a} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n|$$

$$\overline{n^2} = \langle (a^\dagger a)^2 \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} n^2 e^{-\mu n} = \frac{1}{Z} \frac{\partial^2 Z}{\partial \mu^2} = \frac{1}{e^\mu - 1} + \frac{2}{(e^\mu - 1)^2} = \bar{n} + 2\bar{n}^2 \quad (246)$$

$$(\Delta n)^2 = \overline{n^2} - \bar{n}^2 = \bar{n}(\bar{n} + 1)$$

$$\overline{n^k} = \langle (a^\dagger a)^k \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} n^k e^{-\mu n} = \frac{1}{Z} \frac{\partial^k Z}{\partial (-\mu)^k} = \frac{1}{\bar{n} + 1} \left(\bar{n}(\bar{n} + 1) \frac{\partial}{\partial \bar{n}} \right)^k (\bar{n} + 1) \quad (247)$$

$$\overline{n^k} = \frac{1}{Z} \frac{\partial}{\partial (-\mu)} Z \overline{n^{k-1}} = \bar{n} \frac{\partial}{\partial \bar{n}} (\bar{n} + 1) \overline{n^{k-1}} \quad (248)$$

$$S = -\text{tr}(\rho \ln \rho) = \ln Z + \mu \bar{n} = (1 + \bar{n}) \ln(1 + \bar{n}) - \bar{n} \ln \bar{n} \quad (249)$$

$$\begin{aligned} \langle n | \left(\int \frac{d^2 \beta}{\pi \bar{n}} e^{-|\beta|^2 / \bar{n}} |\beta\rangle \langle \beta| \right) | m \rangle &= \int \frac{d^2 \beta}{\pi \bar{n}} e^{-|\beta|^2 / \bar{n}} \langle n | \beta \rangle \langle \beta | m \rangle \\ &= \frac{1}{\sqrt{n! m!}} \int \frac{d^2 \beta}{\pi \bar{n}} e^{-|\beta|^2 (1 + \bar{n}) / \bar{n}} \beta^n (\beta^*)^m \\ &= \frac{1}{2\sqrt{n! m!}} \int \frac{d|\beta|^2 d\phi}{\pi \bar{n}} e^{-|\beta|^2 (1 + \bar{n}) / \bar{n}} |\beta|^{n+m} e^{i(n-m)\phi} \\ &= \frac{1}{n!} \delta_{nm} \int_0^\infty \frac{d|\beta|^2}{\bar{n}} e^{-|\beta|^2 (1 + \bar{n}) / \bar{n}} |\beta|^{2n} \\ &= \frac{1}{n!} \delta_{nm} \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \int_0^\infty du e^{-u} u^n \\ &= \delta_{nm} \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \end{aligned} \quad (250)$$

$$\rho = \int \frac{d^2 \beta}{\pi \bar{n}} e^{-|\beta|^2 / \bar{n}} |\beta\rangle \langle \beta| \implies P(\beta) = W_\rho^{(+1)}(\beta) = \frac{1}{\pi} \tilde{F}_\rho^{(+1)}(\beta) = \frac{1}{\pi \bar{n}} e^{-|\beta|^2 / \bar{n}} \quad (251)$$

$$\Phi_\rho^{(+1)}(\alpha) = F_\rho^{(+1)}(\alpha) = \int d^2 \beta P(\beta) D(\beta, \alpha) = \int \frac{d^2 \beta}{\pi \bar{n}} e^{-|\beta|^2 / \bar{n}} D(\beta, \alpha) = e^{-\bar{n} |\alpha|^2} \quad (252)$$

$$\Phi_\rho^{(s)}(\alpha) = e^{(s-1)|\alpha|^2/2} \Phi_\rho^{(+1)}(\alpha) = e^{-[\bar{n}+(1-s)/2]|\alpha|^2} \quad (253)$$

$$\left\langle [(a^\dagger)^k a^l]_{(s)} \right\rangle = \left. \frac{\partial^{k+l} \Phi_\rho^{(s)}(\alpha)}{\partial \alpha^k \partial (-\alpha^*)^l} \right|_{\alpha=0} = k! \left(\bar{n} + \frac{1-s}{2} \right)^k \delta_{kl} \quad (254)$$

$$\langle : (a^\dagger a)^k : \rangle = \langle (a^\dagger a)^{(k)} \rangle = \langle (a^\dagger)^k a^k \rangle = k! \bar{n}^k \quad (255)$$

$$\begin{aligned} \frac{1}{\pi} \tilde{F}_\rho^{(s)}(\beta) &= W_\rho^{(s)}(\beta) = \int \frac{d^2 \alpha}{\pi^2} \Phi_\rho^{(s)}(\alpha) D(\alpha, \beta) \\ &= \int \frac{d^2 \alpha}{\pi^2} e^{-[\bar{n}+(1-s)/2]|\alpha|^2} D(\alpha, \beta) \\ &= \frac{1}{\pi[\bar{n} + (1-s)/2]} e^{-|\beta|^2/[\bar{n}+(1-s)/2]} \end{aligned} \quad (256)$$

$$\begin{aligned} e^{-\bar{n}|\alpha|^2} &= \Phi_\rho^{(+1)}(\alpha) \\ &= \text{tr}(\rho D^{(+1)}(a, \alpha)) \\ &= \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n \langle n | D^{(+1)}(a, \alpha) | n \rangle \\ &= \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n L_n(|\alpha|^2) \end{aligned} \quad (257)$$

$$e^{-\mu a^\dagger a} = : e^{-(1-e^{-\mu})a^\dagger a} : = : e^{-a^\dagger a/Z} : = : e^{-a^\dagger a/(1+\bar{n})} : \quad (258)$$

$$\langle e^{-\lambda a^\dagger a} \rangle = \langle : e^{(e^{-\lambda}-1)a^\dagger a} : \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} (e^{-\lambda} - 1)^k \langle (a^\dagger)^k a^k \rangle = \sum_{k=0}^{\infty} (\bar{n}(e^{-\lambda} - 1))^k = \frac{1}{1+\bar{n}(1-e^{-\lambda})} \quad (259)$$

$$\langle (1-\lambda)^{a^\dagger a} \rangle = \frac{1}{1+\bar{n}} \sum_{k=0}^{\infty} \left((1-\lambda) \frac{\bar{n}}{1+\bar{n}} \right)^k = \frac{1}{1+\lambda \bar{n}} \quad (260)$$

$$\bar{n}^k = \langle (a^\dagger a)^k \rangle = \left. \frac{\partial^k \langle e^{\lambda a^\dagger a} \rangle}{\partial \lambda^k} \right|_{\lambda=0} = \left. \frac{\partial^k}{\partial \lambda^k} \left(\frac{1}{1-\bar{n}(e^\lambda - 1)} \right) \right|_{\lambda=0} \quad (261)$$

$$\begin{aligned} P(\beta) &= W_\rho^{(+1)}(\beta) = \frac{1}{\pi \bar{n}} e^{-|\beta|^2/\bar{n}} \\ W(\beta) &= W_\rho^{(0)}(\beta) = \frac{1}{\pi(\bar{n} + 1/2)} e^{-|\beta|^2/(\bar{n}+1/2)} \end{aligned} \quad (262)$$

$$Q(\beta) = W_\rho^{(-1)}(\beta) = \frac{1}{\pi} \langle \beta | \rho | \beta \rangle = \frac{1}{\pi(\bar{n} + 1)} \langle \beta | : e^{-a^\dagger a/(\bar{n}+1)} : | \beta \rangle = \frac{1}{\pi(\bar{n} + 1)} e^{-|\beta|^2/(\bar{n}+1)}$$

$$\mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = \int \frac{d^2 \alpha}{\pi} e^{s|\alpha|^2} D^\dagger(a, \alpha) e^{-\mu a^\dagger a} D(a, \alpha) \quad (263)$$

$$\begin{aligned}
\langle \beta | \mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) | \beta \rangle &= \int \frac{d^2 \alpha}{\pi} e^{s|\alpha|^2} \langle \beta | D^\dagger(a, \alpha) e^{-\mu a^\dagger a} D(a, \alpha) | \beta \rangle \quad e^{-\mu} = \frac{\bar{n}}{1 + \bar{n}} \\
&= \int \frac{d^2 \alpha}{\pi} e^{s|\alpha|^2} \underbrace{\langle \beta + \alpha | e^{-\mu a^\dagger a} | \beta + \alpha \rangle}_{= e^{-|\beta + \alpha|^2/(1 + \bar{n})}} \\
&= e^{-|\beta|^2/(1 + \bar{n})} \underbrace{\int \frac{d^2 \alpha}{\pi} \exp\left(-\frac{|\alpha|^2}{1 + \bar{n}}(1 - s(1 + \bar{n}))\right) \exp\left(-\frac{\alpha \beta^* + \alpha^* \beta}{1 + \bar{n}}\right)}_{= \frac{1 + \bar{n}}{1 - s(1 + \bar{n})} \exp\left(\frac{|\beta|^2}{1 + \bar{n}} \frac{1}{1 - s(1 + \bar{n})}\right), \quad s(1 + \bar{n}) \leq 1} \\
&= \frac{1 + \bar{n}}{1 - s(1 + \bar{n})} \underbrace{\exp\left[-|\beta|^2 \left(-\frac{s}{1 - s(1 + \bar{n})}\right)\right]}_{= \langle \beta | e^{-\nu a^\dagger a} | \beta \rangle} \quad e^{-\nu} = \frac{1 - s\bar{n}}{1 - s(1 + \bar{n})} = \frac{\bar{n} - 1/s}{\bar{n} + 1 - 1/s} \\
&= \frac{1 + \bar{n}}{1 - s(1 + \bar{n})} \langle \beta | e^{-\nu a^\dagger a} | \beta \rangle \\
&= -\frac{1}{s} \frac{\bar{n} + 1}{\bar{n} + 1 - 1/s} \langle \beta | e^{-\nu a^\dagger a} | \beta \rangle
\end{aligned} \tag{264}$$

$$\mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = \frac{1 + \bar{n}}{1 - s(1 + \bar{n})} e^{-\nu a^\dagger a} = -\frac{1}{s} \frac{\bar{n} + 1}{\bar{n} + 1 - 1/s} \left(\frac{\bar{n} - 1/s}{\bar{n} + 1 - 1/s} \right)^{a^\dagger a} \tag{265}$$

$$\mathcal{G}^{(s)}\left(\left(\frac{s' + 1}{s' - 1}\right)^{a^\dagger a}\right) = -\frac{1}{s} \frac{1 - s'}{1 - s' - 2/s} \left(\frac{s' + 2/s + 1}{s' + 2/s - 1}\right)^{a^\dagger a} \tag{266}$$

$$\begin{aligned}
s \rightarrow 0^- \quad (\nu = -s): \quad \mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) &= (1 + \bar{n}) e^{s(1 + \bar{n})} e^{sa^\dagger a} \\
\mathcal{G}^{(0)}(e^{-\mu a^\dagger a}) &= (1 + \bar{n}) 1 = Z1
\end{aligned} \tag{267}$$

$$\mathcal{G}^{(-1)}(e^{-\mu a^\dagger a}) = \left(\frac{\bar{n} + 1}{\bar{n} + 2}\right)^{a^\dagger a + 1} \tag{268}$$

$$s \rightarrow -\infty: \quad \mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = -\frac{1}{s} e^{-\mu a^\dagger a} \tag{269}$$

16. Single-mode squeeze operator and single-mode squeezed states

For additional information, see C. M. Caves and B. L. Schumaker, *Physical Review A* **31**, 3068–3092 (1985) [CS]; B. L. Schumaker and C. M. Caves, *Physical Review A* **31**, 3093–3111 (1985) [SC]; and B. L. Schumaker, *Physics Reports* **135**(6), 317–408 (1986) [S].

$$S(r, \phi) \equiv \exp\left(\frac{1}{2}r(a^2e^{-2i\phi} - (a^\dagger)^2e^{2i\phi})\right) = \exp\left(i\frac{1}{2}r[(xp + px)\cos 2\phi - (x^2 - p^2)\sin 2\phi]\right) \quad (270)$$

$$S(r, \phi + \pi) = S(r, \phi) \quad (271)$$

$$S^{-1}(r, \phi) = S^\dagger(r, \phi) = S(-r, \phi) = S(r, \phi + \pi/2) \quad (272)$$

$$e^{i\theta a^\dagger a} S(r, \phi) e^{-i\theta a^\dagger a} = S(r, \phi + \theta) \quad (273)$$

$$\begin{aligned} S(r, \phi) &= e^{-\Gamma A^\dagger} e^{-gB} e^{\Gamma^* A} = e^{-\Gamma A^\dagger} e^{\Gamma^* e^{2g} A} e^{-gB} = e^{-gB} e^{-\Gamma e^{2g} A^\dagger} e^{\Gamma^* A} \\ &= e^{\Gamma^* A} e^{-\Gamma e^{2g} A^\dagger} e^{gB} = e^{\Gamma^* A} e^{gB} e^{-\Gamma A^\dagger} = e^{gB} e^{\Gamma^* e^{2g} A} e^{-\Gamma A^\dagger} \end{aligned} \quad (274)$$

$$A \equiv \frac{1}{2}a^2 \quad B \equiv a^\dagger a + \frac{1}{2} \quad \Gamma \equiv e^{2i\phi} \tanh r \quad g \equiv \ln(\cosh r)$$

(See Appendix B of [SC].)

$$\begin{aligned} S^\dagger(r, \phi) a S(r, \phi) &= a \cosh r - a^\dagger e^{2i\phi} \sinh r \\ S^\dagger(r, \phi) a^\dagger S(r, \phi) &= a^\dagger \cosh r - a e^{-2i\phi} \sinh r \end{aligned} \quad (275)$$

$$S^\dagger(r, \phi) D(a, \alpha) S(r, \phi) = D(a \cosh r - a^\dagger e^{2i\phi} \sinh r, \alpha) = D(a, \alpha \cosh r + \alpha^* e^{2i\phi} \sinh r) \quad (276)$$

$$\begin{aligned} S^\dagger(r, \phi) (x \cos \phi + p \sin \phi) S(r, \phi) &= (x \cos \phi + p \sin \phi) e^{-r} \\ S^\dagger(r, \phi) (-x \sin \phi + p \cos \phi) S(r, \phi) &= (-x \sin \phi + p \cos \phi) e^r \end{aligned} \quad (277)$$

$$S^\dagger(r, 0) x S(r, 0) = x e^{-r} \quad S^\dagger(r, 0) p S(r, 0) = p e^r \quad (278)$$

$$\begin{aligned} S^\dagger(r', \phi') S(r, \phi) &= e^{-i\Theta B} S(R, \Phi) = S(R, \Phi - \Theta) e^{-i\Theta B} \\ B &\equiv a^\dagger a + \frac{1}{2} \end{aligned} \quad (279)$$

$$\begin{aligned} e^{i\Theta} \cosh R &\equiv \cosh r \cosh r' - e^{2i(\phi - \phi')} \sinh r \sinh r' \\ e^{i(2\Phi - \Theta)} \sinh R &\equiv e^{2i\phi} \sinh r \cosh r' - e^{2i\phi'} \cosh r \sinh r' \end{aligned}$$

(See Appendix B of [SC].)

$$S(r', \phi) S(r, \phi) = S(r + r', \phi) \quad (280)$$

$$\begin{aligned} |\alpha\rangle_{(r, \phi)} &\equiv D(a, \alpha) S(r, \phi) |0\rangle \\ &= D(a, \alpha) |0\rangle_{(r, \phi)} \\ &= S(r, \phi) D(a, \alpha \cosh r + \alpha^* e^{2i\phi} \sinh r) |0\rangle \\ &= S(r, \phi) |\alpha \cosh r + \alpha^* e^{2i\phi} \sinh r\rangle \end{aligned} \quad (281)$$

$$e^{-i\theta a^\dagger a} |\alpha\rangle_{(r, \phi)} = |\alpha e^{-i\theta}\rangle_{(r, \phi - \theta)} \quad (282)$$

$$\begin{aligned} (a \cosh r + a^\dagger e^{2i\phi} \sinh r) |\alpha\rangle_{(r, \phi)} &= S(r, \phi) a |\alpha \cosh r + \alpha^* e^{2i\phi} \sinh r\rangle \\ &= (\alpha \cosh r + \alpha^* e^{2i\phi} \sinh r) |\alpha\rangle_{(r, \phi)} \end{aligned} \quad (283)$$

$$S^\dagger(r, \phi) D^\dagger(a, \alpha) a D(a, \alpha) S(r, \phi) = a \cosh r - a^\dagger e^{2i\phi} \sinh r + \alpha \quad (284)$$

$$\begin{aligned}
S^\dagger(r, \phi) D^\dagger(a, \gamma) D(a, \alpha) D(a, \gamma) S(r, \phi) &= D(a \cosh r - a^\dagger e^{2i\phi} \sinh r + \gamma, \alpha) \\
&= D(\gamma, \alpha) D(a, \alpha \cosh r + \alpha^* e^{2i\phi} \sinh r)
\end{aligned} \tag{285}$$

$$\begin{aligned}
{}_{(r, \phi)} \langle \alpha | a | \alpha \rangle_{(r, \phi)} &= \alpha \\
{}_{(r, \phi)} \langle \alpha | a^2 | \alpha \rangle_{(r, \phi)} &= \alpha^2 - e^{2i\phi} \cosh r \sinh r = \alpha^2 - \frac{1}{2} e^{2i\phi} \sinh 2r \\
{}_{(r, \phi)} \langle \alpha | a^\dagger a | \alpha \rangle_{(r, \phi)} &= |\alpha|^2 + \sinh^2 r = |\alpha|^2 + \frac{1}{2} (\cosh 2r - 1)
\end{aligned} \tag{286}$$

$$\begin{aligned}
{}_{(r, \phi)} \langle \alpha | x | \alpha \rangle_{(r, \phi)} &= \alpha_1 \\
{}_{(r, \phi)} \langle \alpha | p | \alpha \rangle_{(r, \phi)} &= \alpha_2 \\
{}_{(r, \phi)} \langle \alpha | (\Delta x)^2 | \alpha \rangle_{(r, \phi)} &= \frac{1}{2} (\cosh 2r - \sinh 2r \cos 2\phi) = \frac{1}{2} (e^{-2r} \cos^2 \phi + e^{2r} \sin^2 \phi) \\
{}_{(r, \phi)} \langle \alpha | (\Delta p)^2 | \alpha \rangle_{(r, \phi)} &= \frac{1}{2} (\cosh 2r + \sinh 2r \cos 2\phi) = \frac{1}{2} (e^{-2r} \sin^2 \phi + e^{2r} \cos^2 \phi) \\
{}_{(r, \phi)} \langle \alpha | \frac{1}{2} (\Delta x \Delta p + \Delta p \Delta x) | \alpha \rangle_{(r, \phi)} &= -\frac{1}{2} \sinh 2r \sin 2\phi
\end{aligned} \tag{287}$$

$$\begin{aligned}
|0\rangle_{(r, \phi)} &= S(r, \phi) |0\rangle = \frac{1}{\sqrt{\cosh r}} \exp\left(-\frac{1}{2} (a^\dagger)^2 e^{2i\phi} \tanh r\right) |0\rangle \\
&= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2} e^{2i\phi} \tanh r)^n}{n!} \underbrace{(a^\dagger)^{2n} |0\rangle}_{= \sqrt{(2n)!} |2n\rangle} \\
&= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \left(-\frac{1}{\sqrt{2}} e^{2i\phi} \tanh r\right)^n \sqrt{\frac{(2n-1)!!}{n!}} |2n\rangle \\
&\quad [(2n)! = 2^n n! (2n-1)!!]
\end{aligned} \tag{288}$$

$${}_{(r, \phi)} \langle 0 | 0 \rangle_{(r, \phi)} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \left(\frac{\tanh^2 r}{2}\right)^n \frac{(2n-1)!!}{n!} = \frac{1}{\cosh r \sqrt{1 - \tanh^2 r}} = 1 \tag{289}$$

$$\begin{aligned}
&\left(\text{Use } (1+x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} \left(\frac{-x}{2}\right)^n \right. \\
&{}_{(r, \phi)} \langle 0 | a^\dagger a | 0 \rangle_{(r, \phi)} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \left(\frac{\tanh^2 r}{2}\right)^n 2n \frac{(2n-1)!!}{n!} \\
&= \frac{\sinh^2 r}{\cosh^3 r} \sum_{n=0}^{\infty} \left(\frac{\tanh^2 r}{2}\right)^n \frac{(2n+1)!!}{n!} \\
&= \frac{\sinh^2 r}{\cosh^3 r} \frac{1}{(1 - \tanh^2 r)^{3/2}} \\
&= \sinh^2 r \\
&\left(\text{Use } (1+x)^{-3/2} = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{n!} \left(\frac{-x}{2}\right)^n \right.
\end{aligned} \tag{290}$$

$$\begin{aligned}
{}_{(r,\phi)}\langle 0|a^\dagger a(a^\dagger a - 2)|0\rangle_{(r,\phi)} &= \frac{1}{\cosh r} \sum_{n=0}^{\infty} \left(\frac{\tanh^2 r}{2} \right)^n 4n(n-1) \frac{(2n-1)!!}{n!} \\
&= \frac{\sinh^4 r}{\cosh^5 r} \sum_{n=0}^{\infty} \left(\frac{\tanh^2 r}{2} \right)^n \frac{(2n+3)!!}{n!} \\
&= \frac{\sinh^4 r}{\cosh^5 r} \frac{3}{(1 - \tanh^2 r)^{5/2}} \\
&= 3 \sinh^4 r
\end{aligned} \tag{291}$$

$$\left(\text{Use } (1+x)^{-5/2} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(2n+3)!!}{n!} \left(\frac{-x}{2} \right)^n \right)$$

$${}_{(r,\phi)}\langle 0|(a^\dagger a)^2|0\rangle_{(r,\phi)} = 3 \sinh^4 r + 2 \sinh^2 r \tag{292}$$

$${}_{(r,\phi)}\langle 0|(\Delta a^\dagger a)^2|0\rangle_{(r,\phi)} = 2 \sinh^4 r + 2 \sinh^2 r = 2 \sinh^2 r \cosh^2 r \tag{293}$$

$$\begin{aligned}
\langle x=0|0\rangle_{(r,\phi)} &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}e^{2i\phi} \tanh r)^n}{n!} \underbrace{\sqrt{(2n)!} \langle x=0|2n\rangle}_{= \frac{1}{\pi^{1/4} 2^n} H_{2n}(0)} \\
&= \frac{1}{\pi^{1/4} \sqrt{\cosh r}} \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{2i\phi} \tanh r \right)^n \frac{(2n-1)!!}{n!} \\
&= \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi} \sinh r)^{1/2}}
\end{aligned} \tag{294}$$

$$[\text{Use } H_{2n}(0) = (-1)^n (2n)!/n! = (-2)^n (2n-1)!! : \text{A\&S 22.3.10.}]$$

$$\begin{aligned}
0 &= \langle x|(a \cosh r + a^\dagger e^{2i\phi} \sinh r)|0\rangle_{(r,\phi)} \\
&= \frac{1}{\sqrt{2}} \left((\cosh r + e^{2i\phi} \sinh r)x + (\cosh r - e^{2i\phi} \sinh r) \frac{d}{dx} \right) \langle x|0\rangle_{(r,\phi)} \\
\langle x|0\rangle_{(r,\phi)} &= \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi} \sinh r)^{1/2}} \exp \left(-\frac{1}{2} \frac{\cosh r + e^{2i\phi} \sinh r}{\cosh r - e^{2i\phi} \sinh r} x^2 \right) \\
\Rightarrow &= \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi} \sinh r)^{1/2}} \exp \left(-\frac{1}{2} \frac{1 + i \sinh 2r \sin 2\phi}{\cosh 2r - \sinh 2r \cos 2\phi} x^2 \right)
\end{aligned} \tag{295}$$

$$\begin{aligned}
\langle x|\alpha\rangle_{(r,\phi)} &= \langle x|D(a, \alpha)|0\rangle_{(r,\phi)} \\
&= e^{-i\alpha_1 \alpha_2/2} e^{i\alpha_2 x} \langle x - \alpha_1|0\rangle_{(r,\phi)} \\
&= \frac{1}{\pi^{1/4}} \frac{e^{-i\alpha_1 \alpha_2/2} e^{i\alpha_2 x}}{(\cosh r - e^{2i\phi} \sinh r)^{1/2}} \exp \left(-\frac{1}{2} \frac{1 + i \sinh 2r \sin 2\phi}{\cosh 2r - \sinh 2r \cos 2\phi} (x - \alpha_1)^2 \right)
\end{aligned} \tag{296}$$

$$\begin{aligned}
\langle \beta|0\rangle_{(r,\phi)} &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}e^{2i\phi} \tanh r)^n}{n!} \underbrace{\sqrt{(2n)!} \langle \beta|2n\rangle}_{= e^{-|\beta|^2/2} (\beta^*)^{2n}} \\
&= \frac{1}{\sqrt{\cosh r}} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}(\beta^* e^{i\phi})^2 \tanh r)^n}{n!} \\
&= \frac{1}{\sqrt{\cosh r}} e^{-|\beta|^2/2} \exp \left(-\frac{1}{2} (\beta^* e^{i\phi})^2 \tanh r \right)
\end{aligned} \tag{297}$$

$$\begin{aligned}
\langle \beta | \alpha \rangle_{(r,\phi)} &= \langle \beta | D(a, \alpha) | 0 \rangle_{(r,\phi)} \\
&= D(\beta, \alpha/2) \langle \beta - \alpha | 0 \rangle_{(r,\phi)} \\
&= \frac{1}{\sqrt{\cosh r}} D(\beta, \alpha/2) e^{-|\beta - \alpha|^2/2} \exp\left(-\frac{1}{2}[(\beta^* - \alpha^*)e^{i\phi}]^2 \tanh r\right)
\end{aligned} \tag{298}$$

$$\begin{aligned}
|\langle \beta | \alpha \rangle_{(r,\phi)}|^2 &= |\langle \beta - \alpha | 0 \rangle_{(r,\phi)}|^2 \\
&= \frac{1}{\cosh r} \exp\left[-\left(|\beta - \alpha|^2 + \frac{1}{2}\left([\beta - \alpha]e^{-i\phi}]^2 + [(\beta^* - \alpha^*)e^{i\phi}]^2\right) \tanh r\right)\right] \\
&= \frac{1}{\cosh r} \exp\left[-\frac{1}{2}\left((\beta_1 - \alpha_1)^2(1 + \tanh r \cos 2\phi) \right. \right. \\
&\quad \left. \left. + (\beta_2 - \alpha_2)^2(1 - \tanh r \cos 2\phi) \right. \right. \\
&\quad \left. \left. + 2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \tanh r \sin 2\phi\right)\right]
\end{aligned} \tag{299}$$

$$\rho_{\alpha;r,\phi} \equiv |\alpha\rangle_{(r,\phi)(r,\phi)} \langle \alpha| \tag{300}$$

$$\begin{aligned}
\Phi_{\rho_{\gamma;r,\phi}}^{(s)}(\alpha) &= {}_{(r,\phi)}\langle \gamma | D^{(s)}(a, \alpha) | \gamma \rangle_{(r,\phi)} \\
&= e^{s|\alpha|^2/2} \langle 0 | S^\dagger(r, \phi) D^\dagger(a, \gamma) D(a, \alpha) D(a, \gamma) S(r, \phi) | 0 \rangle \\
&= D(\gamma, \alpha) e^{s|\alpha|^2/2} \langle 0 | D(a, \alpha \cosh r + \alpha^* e^{2i\phi} \sinh r) | 0 \rangle \\
&= D(\gamma, \alpha) e^{s|\alpha|^2/2} \exp\left(-\frac{1}{2}|\alpha \cosh r + \alpha^* e^{2i\phi} \sinh r|^2\right) \\
&= D(\gamma, \alpha) \exp\left[-\frac{1}{2}\left(|\alpha|^2(-s + \cosh 2r) + \frac{1}{2}[(\alpha e^{-i\phi})^2 + (\alpha^* e^{i\phi})^2] \sinh 2r\right)\right] \\
&= D(\gamma, \alpha) \exp\left[-\frac{1}{4}\left(\alpha_1^2(-s + \cosh 2r + \sinh 2r \cos 2\phi) \right. \right. \\
&\quad \left. \left. + \alpha_2^2(-s + \cosh 2r - \sinh 2r \cos 2\phi) \right. \right. \\
&\quad \left. \left. + 2\alpha_1\alpha_2 \sinh 2r \sin 2\phi\right)\right] \\
&= D(\gamma, \alpha) \exp\left[-\frac{1}{4}\left(\alpha_1^2(-s + e^{-2r} \sin^2 \phi + e^{2r} \cos^2 \phi) \right. \right. \\
&\quad \left. \left. + \alpha_2^2(-s + e^{-2r} \cos^2 \phi + e^{2r} \sin^2 \phi) \right. \right. \\
&\quad \left. \left. + 2\alpha_1\alpha_2 \sinh 2r \sin 2\phi\right)\right]
\end{aligned} \tag{301}$$

$$\begin{aligned}
W_{\rho_{0;r,0}}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2} \Phi_{\rho_{0;r,0}}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \frac{1}{\pi} \int \frac{d\alpha_1 d\alpha_2}{2\pi} e^{i(\beta_2\alpha_1 - \beta_1\alpha_2)} e^{-[\alpha_1^2(-s + e^{2r}) + \alpha_2^2(-s + e^{-2r})]/4} \\
&= \frac{2}{\pi} \frac{1}{\sqrt{(-s + e^{-2r})(-s + e^{2r})}} \exp\left(-\frac{\beta_1^2}{-s + e^{-2r}} - \frac{\beta_2^2}{-s + e^{2r}}\right) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1 - 2s \cosh 2r + s^2}} \exp\left(-\frac{2|\beta|^2(\cosh 2r - s) + [\beta^2 + (\beta^*)^2] \sinh 2r}{1 - 2s \cosh 2r + s^2}\right),
\end{aligned} \tag{302}$$

$s \leq e^{-2|r|}$

$$\begin{aligned}
W_{\rho_0;r,\phi}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2} \Phi_{\rho_0;r,\phi}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi^2} \underbrace{\Phi_{\rho_0;r,\phi}^{(s)}(\alpha e^{i\phi})}_{=\Phi_{\rho_0;r,0}^{(s)}(\alpha)} D(\alpha, \beta e^{-i\phi}) \\
&= W_{\rho_0;r,0}^{(s)}(\beta e^{-i\phi}) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1-2s \cosh 2r + s^2}} \\
&\quad \times \exp\left(-\frac{2|\beta|^2(\cosh 2r - s) + [(\beta e^{-i\phi})^2 + (\beta^* e^{i\phi})^2] \sinh 2r}{1-2s \cosh 2r + s^2}\right) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1-2s \cosh 2r + s^2}} \\
&\quad \times \exp\left(\frac{\begin{aligned} &-\beta_1^2(-s + \cosh 2r + \sinh 2r \cos 2\phi) \\ &-\beta_2^2(-s + \cosh 2r - \sinh 2r \cos 2\phi) \\ &-2\beta_1\beta_2 \sinh 2r \sin 2\phi \end{aligned}}{1-2s \cosh 2r + s^2}\right), \quad s \leq e^{-2|r|}
\end{aligned} \tag{303}$$

$$\begin{aligned}
W_{\rho_\gamma;r,\phi}^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2} \Phi_{\rho_\gamma;r,\phi}^{(s)}(\alpha) D(\alpha, \beta) \\
&= \int \frac{d^2\alpha}{\pi^2} \Phi_{\rho_0;r,\phi}^{(s)}(\alpha) D(\alpha, \beta - \gamma) \\
&= W_{\rho_0;r,\phi}^{(s)}(\beta - \gamma) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1-2s \cosh 2r + s^2}} \\
&\quad \times \exp\left(-\frac{2|\beta - \gamma|^2(\cosh 2r - s) + [(\beta - \gamma)e^{-i\phi}]^2 + [(\beta^* - \gamma^*)e^{i\phi}]^2 \sinh 2r}{1-2s \cosh 2r + s^2}\right) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{1-2s \cosh 2r + s^2}} \\
&\quad \times \exp\left(\frac{\begin{aligned} &-(\beta_1 - \gamma_1)^2(-s + \cosh 2r + \sinh 2r \cos 2\phi) \\ &-(\beta_2 - \gamma_2)^2(-s + \cosh 2r - \sinh 2r \cos 2\phi) \\ &-2(\beta_1 - \gamma_1)(\beta_2 - \gamma_2) \sinh 2r \sin 2\phi \end{aligned}}{1-2s \cosh 2r + s^2}\right), \quad s \leq e^{-2|r|}
\end{aligned} \tag{304}$$

$$\rho = \rho_{\gamma;r,\phi} : \left\{ \begin{array}{ll} s = 0: & \begin{aligned} W(\beta) &= \frac{2}{\pi} e^{-2|\beta-\gamma|^2 \cosh 2r - ([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2) \sinh 2r} \\ W'(\beta_1, \beta_2) &= \frac{1}{\pi} \exp \left(\begin{array}{l} -(\beta_1-\gamma_1)^2 (\cosh 2r + \sinh 2r \cos 2\phi) \\ -(\beta_2-\gamma_2)^2 (\cosh 2r - \sinh 2r \cos 2\phi) \\ -2(\beta_1-\gamma_1)(\beta_2-\gamma_2) \sinh 2r \sin 2\phi \end{array} \right) \\ Q(\beta) &= \frac{1}{\pi \cosh r} e^{-|\beta-\gamma|^2 - \frac{1}{2}([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2) \tanh r} \end{aligned} \\ s = -1: & \begin{aligned} &= \frac{1}{\pi \cosh r} \exp \left(\begin{array}{l} -\frac{1}{2}(\beta_1-\gamma_1)^2 (1 + \tanh r \cos 2\phi) \\ -\frac{1}{2}(\beta_2-\gamma_2)^2 (1 - \tanh r \cos 2\phi) \\ -(\beta_1-\gamma_1)(\beta_2-\gamma_2) \tanh r \sin 2\phi \end{array} \right) \\ &= \frac{1}{\pi} |\langle \beta | \gamma \rangle_{(r,\phi)}|^2 \end{aligned} \end{array} \right. \quad (305)$$

$$\begin{aligned} \rho = \rho_{0;r,0} : \quad W(\beta) &= \frac{2}{\pi} {}_{(r,0)}\langle 0 | PD(a, -2\beta) | 0 \rangle_{(r,\phi)} \\ &= \frac{2}{\pi} {}_{(r,0)}\langle 0 | D(a, -2\beta) | 0 \rangle_{(r,\phi)} \\ &= \frac{2}{\pi} \Phi_{\rho_{0;r,0}}^{(0)}(-2\beta) \\ &= \frac{2}{\pi} e^{-2|\beta|^2 \cosh 2r - [\beta^2 + (\beta^*)^2] \sinh 2r} \end{aligned} \quad (306)$$

17. Auxiliary formulae

$$\frac{d^k f(x)g(x)}{dx^k} = \sum_{m=0}^k \frac{k!}{m!(k-m)!} \frac{d^m f(x)}{dx^m} \frac{d^{k-m} g(x)}{dx^{k-m}} \quad (307)$$

$$\frac{\partial^{k+l} f(x,y)g(x,y)}{\partial x^k \partial y^l} = \sum_{m=0}^k \sum_{n=0}^l \frac{k!}{m!(k-m)!} \frac{l!}{n!(l-n)!} \frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n} \frac{\partial^{k+l-m-n} g(x,y)}{\partial x^{k-m} \partial y^{l-n}} \quad (308)$$

$$f(xy) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (xy)^k = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{\partial^{2k} f(xy)}{\partial x^k \partial y^k} \Big|_{x=y=0} (xy)^k \quad (309)$$

$$\frac{\partial (xy)^k}{\partial x \partial y} = k^2 (xy)^{k-1} = k \frac{d(xy)^k}{d(xy)} \quad (310)$$

$$\frac{\partial^{2k} f(xy)}{\partial x^k \partial y^k} \Big|_{x=y=0} = k! f^{(k)}(0) \quad (311)$$